(1) The energy levels of the hydrogen atom are given by the famous Bohr formula

\[ E_n = -\left(\frac{m_e}{2\hbar^2}\right) \left(\frac{e^2}{4\pi\epsilon_0}\right)^2 \frac{1}{n^2} \approx -13.6 \text{ eV} \]

for \( n = 1, 2, 3, \ldots \), and the characteristic size of the atom is given by the Bohr radius

\[ a = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} \approx 0.5 \times 10^{-10} \text{ m}. \]

Now consider a hydrogenic atom, which consists of a single electron orbiting a nucleus with \( Z \) protons (\( Z = 1 \) is hydrogen itself, \( Z = 2 \) is ionized helium, \( Z = 3 \) is doubly ionized lithium, and so on). Determine the Bohr energies \( E_n(Z) \) and the Bohr radius \( a(Z) \) for a hydrogenic atom (you can write them in terms of \( E_n \) and \( a \) if you like). Use the concept of reduced mass to show why the mass of the nucleus can be neglected to good approximation.

(2) The angular momentum operators satisfy the commutation relations

\[ [L_x, L_y] = i\hbar L_z, \quad [L_y, L_z] = i\hbar L_x, \quad [L_z, L_x] = i\hbar L_y, \]

but \( L^2 \equiv L_x^2 + L_y^2 + L_z^2 \) commutes with those same operators:

\[ [L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0. \]

The eigenfunctions of \( L^2 \) and \( L_z \) \((f_m^\ell)\) are characterized by the numbers \( \ell \) and \( m \) such that

\[ L^2 f_m^\ell = \hbar^2 \ell(\ell + 1)f_m^\ell; \quad L_z f_m^\ell = \hbar m f_m^\ell. \]

We also defined the raising and lowering operators

\[ L_\pm \equiv L_x \pm iL_y, \]

such that

\[ L_\pm f_m^\ell = \hbar \sqrt{\ell(\ell + 1) - m(m \pm 1)} f_{m \pm 1}^\ell. \]

(a) What is \( L_+ Y_\ell^m \)? (No calculation is allowed!)

(b) Use the above result together with the fact that \( L_z Y_\ell^m = \hbar \ell Y_\ell^m \) to determine \( Y_\ell^m(\theta, \phi) \) up to a normalization constant. Hint: you can determine what \( L_+ \) and \( L_z \) are in spherical coordinates from our in-class results:

\[ L_z = \frac{\hbar}{i} \frac{\partial}{\partial \phi}, \]

\[ L_+ = \hbar e^{i\phi} \left( \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \phi} \right). \]
(c) Determine the normalization constant of $Y_{\ell}^m$ by direct integration (by parts). Compare your result to the answer for question 2b in HW7.

(d) In question 2a of HW7, you showed that

$$Y_2^1(\theta, \phi) = -\frac{1}{2}\sqrt{15}\frac{\sin \theta \cos \theta e^{i\phi}}{2\pi}.$$ 

Use the raising operator to compute $Y_2^2(\theta, \phi)$. Use the fact that $L_\pm f^m_\ell = \hbar \sqrt{\ell(\ell + 1)} - m(m \pm 1) f^{m \pm 1}_\ell$ to get the correct normalization.

(3) Two particles of mass $m$ are attached to the ends of a massless rigid rod of length $a$. The system is free to rotate in three dimensions about the center (but the center point itself is fixed).

(a) Show that the allowed energies of this rigid rotor are

$$E_n = \frac{\hbar^2 n(n + 1)}{ma^2}$$

for $n = 0, 1, 2, \ldots$. You will want to start this problem by expressing the classical energy in terms of the total angular momentum.

(b) What are the normalized eigenfunctions $\psi$ for this system? What is the degeneracy $d(n)$ of the $n$th energy level? Hint: you don’t need a lot of novel calculations to solve this problem.

(4) Consider the three-dimensional harmonic oscillator, for which the potential is

$$V(r) = \frac{1}{2}m \omega^2 r^2.$$ 

(a) Show that separation of variables in Cartesian coordinates turns this into three one-dimensional oscillators, and exploit your knowledge of the latter to show that the allowed energies are

$$E_n = \left(n + \frac{3}{2}\right) \hbar \omega.$$ 

(b) Determine the degeneracy $d(n)$ of $E_n$.

(5) Suppose the electron is a classical solid sphere with radius

$$r_c = \frac{e^2}{4\pi \epsilon_0 mc^2}.$$ 

This is the so-called “classical electron radius,” obtained by assuming that the electron’s mass is attributable to energy stored in its electric field, via $E = mc^2$. Given also that the electron’s angular momentum is $\frac{1}{2} \hbar$, how fast (in m/s) is a point on the “equator” moving? Does this model for spin make sense? Experimentally, the electron’s radius is known to be much smaller, but that only makes matters worse.