1. Question 1

a) \[
[x, p_x] f = \frac{\hbar}{i} \left( x \frac{\partial f}{\partial x} - \frac{\partial}{\partial x} (xf) \right) = \frac{\hbar}{i} \left( x \frac{\partial f}{\partial x} - f - x \frac{\partial f}{\partial x} \right) = -\frac{\hbar}{i} f,
\]
so \([x, p_x] = i\hbar\), and similarly, \([y, p_y] = i\hbar\), and \([z, p_z] = i\hbar\). Furthermore,
\[
[x, p_y] f = \frac{\hbar}{i} \left( x \frac{\partial f}{\partial y} - \frac{\partial}{\partial y} (xf) \right) = \frac{\hbar}{i} \left( x \frac{\partial f}{\partial y} - x \frac{\partial f}{\partial y} \right) = 0,
\]
as well as all other permutations where for \(r_i\) and \(p_j\) where \(i \neq j\). In general,
\[
-[A, B] = -(AB - BA) = BA - AB = [B, A]
\]so
\[
[r_i, p_j] = [p_i, r_j] = i\hbar \delta_{ij}.
\]
Similarly, \([x, y] f = (xy - yx) f = 0\). Also,
\[
[p_x, p_y] f = -\hbar^2 \left( \frac{\partial}{\partial x} \frac{\partial}{\partial y} - \frac{\partial}{\partial y} \frac{\partial}{\partial x} \right) f = -\hbar^2 \left( \frac{\partial^2}{\partial x \partial y} - \frac{\partial^2}{\partial y \partial x} \right) f = 0.
\]
Since this is true for all permutations of \(i\) and \(j\),
\[
[r_i, r_j] = [p_i, p_j] = 0.
\]

b) Using separation of variables, we define \(\psi(x, y, z) = X(x)Y(y)Z(z)\), so applying this to the Schrödinger equation and dividing by \(XYZ\) we get
\[
\frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} = -\frac{2m}{\hbar^2} E.
\]
Let the separation constants be \(k_x^2, k_y^2,\) and \(k_z^2\), such that
\[
E = \frac{\hbar^2}{2m} \left( k_x^2 + k_y^2 + k_z^2 \right),
\]
and
\[
\frac{d^2 X}{dx^2} = -k_x^2 X; \quad \frac{d^2 Y}{dy^2} = -k_y^2 Y; \quad \frac{d^2 Z}{dz^2} = -k_z^2 Z.
\]
The solution for the \(x\) component is
\[
X(x) = A \sin(k_x x) + B \cos(k_x x),
\]
but \( X(0) = 0 \), so \( B = 0 \), and \( X(a) = 0 \), so \( k_x a = n_x \pi \), where \( n_x = 1, 2, 3, \ldots \), where (as we worked out in class) we have dropped the \( n_x = 0 \) case because it results in a non-normalizable solution and the negative solutions are absorbed into the normalization constant. Finally, the normalization condition requires that \( A_x = \sqrt{2/a} \). By symmetry, this applies to the \( y \) and \( z \) coordinates as well, so that

\[
\psi(x, y, z) = \left( \frac{2}{a} \right)^{3/2} \sin \left( \frac{n_x \pi}{a} x \right) \sin \left( \frac{n_y \pi}{a} y \right) \sin \left( \frac{n_z \pi}{a} z \right),
\]

where \( n_x, n_y, \) and \( n_z \) are all positive integers. Since \( k_i = n_i \pi/a \), the corresponding energies are

\[
E = \frac{\hbar^2 \pi^2}{2ma^2} (n_x^2 + n_y^2 + n_z^2).
\]

c) Using the notation \( E_n(n_x, n_y, n_z) \), and \( E_0 \equiv \frac{\hbar^2 \pi^2}{2ma^2} \), we have

\[
\begin{align*}
E_1(1, 1, 1) &= 3E_0 \\
E_2(2, 1, 1) &= 6E_0 \\
E_2(1, 2, 1) &= 6E_0 \\
E_2(1, 1, 2) &= 6E_0 \\
E_3(1, 2, 2) &= 9E_0 \\
E_3(2, 1, 2) &= 9E_0 \\
E_3(2, 2, 1) &= 9E_0 \\
E_4(3, 1, 1) &= 11E_0 \\
E_4(1, 3, 1) &= 11E_0 \\
E_4(1, 1, 3) &= 11E_0 \\
E_5(2, 2, 2) &= 12E_0 \\
E_6(1, 2, 3) &= 14E_0 \\
E_6(3, 1, 2) &= 14E_0 \\
E_6(2, 3, 1) &= 14E_0 \\
E_6(1, 3, 2) &= 14E_0 \\
E_6(2, 1, 3) &= 14E_0 \\
E_6(3, 2, 1) &= 14E_0.
\end{align*}
\]

Reading off the results, the number of degeneracies corresponding to \( E_1 \) to \( E_6 \) are: 1, 3, 3, 1, 6, respectively.

d) The next set of energies (suppressing the permutations) are \( E_7(3, 2, 2), E_8(4, 1, 1), E_9(3, 3, 1), E_{10}(4, 2, 1), E_{11}(3, 3, 2), E_{12}(4, 2, 2), E_{14}(4, 3, 1), \) and \( E_{14}(3, 3, 3) \) as well as \( E_{14}(5, 1, 1) \). The number of degeneracies correspond to the possible permutations: 1 if
\(n_x = n_y = n_z\), 3 if there is one unique \(n_i\), and 6 if for all three \(n_x \neq n_y \neq n_z\). So for \(E_{14}\), there are \(1 + 3 = 4\) degeneracies.

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2. **Question 2**

a) \(P_0(x) = 1\), and \(P_0^0(x) = 1\), so

\[
Y_0^0(\theta, \phi) = 1 \cdot \sqrt{\frac{(2 \cdot 0 + 1)(0 - 0)!}{4\pi}} e^{i \theta} = 1
\]

To calculate \(Y_2^1\), we have

\[
P_2(x) = \frac{1}{2^2 \cdot 2!} \left( \frac{d}{dx} \right)^2 (x^2 - 1)^2 = \frac{1}{8 \cdot 2!} (x^4 - 2x^2 + 1) = \frac{1}{2} (3x^2 - 1),
\]

\[
P_2^1(x) = \sqrt{1 - x^2} \frac{d}{dx} P_2(x) = 3x \sqrt{1 - x^2} \rightarrow P_2^1(\cos \theta) = 3 \cos \theta \sqrt{1 - \cos^2 \theta} = 3 \cos \theta \sin \theta.
\]

Therefore

\[
Y_2^1(\theta, \phi) = (-1)^1 \sqrt{\frac{(2 \cdot 2 + 1)(2 - 1)!}{4\pi}} e^{i \phi} P_2^1(\cos \theta) = \frac{1}{2} \sqrt{\frac{15}{2\pi}} \sin \theta \cos \theta e^{i \phi}.
\]

To show the normalization of \(Y_0^0\), we take

\[
\int_0^\pi \int_0^{2\pi} |Y_0^0|^2 \sin \theta d\theta d\phi = \frac{1}{4\pi} (-\cos \theta) \bigg|_0^{2\pi} = 1.
\]

To show the normalization of \(Y_2^1\), we take

\[
\int_0^\pi \int_0^{2\pi} |Y_2^1|^2 \sin \theta d\theta d\phi = \frac{15}{8\pi} \int_0^\pi \int_0^{2\pi} \sin^3 \theta \cos^2 \theta d\theta d\phi
\]

\[
= \frac{15}{4} \int_0^\pi \sin \theta (\cos^2 \theta - \cos^4 \theta) d\theta
\]

\[
= \frac{15}{4} \left[ -\frac{1}{3} \cos^3 \theta + \frac{1}{5} \cos^5 \theta \right]_0^\pi
\]

\[
= \frac{15}{4} \left( \frac{2}{3} - \frac{2}{5} \right) = 1.
\]

To show the orthogonality of \(Y_0^0\) and \(Y_2^1\), we take

\[
\int_0^\pi \int_0^{2\pi} (Y_0^0)^* Y_2^1 \sin \theta d\theta d\phi = \frac{-1}{4\pi} \sqrt{\frac{15}{2}} \int_0^\pi \int_0^{2\pi} \sin^2 \theta \cos \theta d\theta e^{i \phi} d\phi
\]

\[
= \frac{-1}{4\pi} \sqrt{\frac{15}{2}} \sin^2 \theta \bigg|_0^{2\pi} = 0.
\]
since the \( \sin^3 \theta \) term vanishes when evaluated at 0 and \( \pi \).

b) When \( m = \ell \), \( P_\ell^m \) involves \( 2\ell \) derivatives of a \( 2\ell \)-th order polynomial. All terms of lower order than \( 2\ell \) go to 0. Therefore we can write it as

\[
P_\ell^m(x) = (1 - x^2)^{\ell/2} \frac{1}{2\ell!} \frac{d^{2\ell}}{dx^{2\ell}} x^{2\ell} = (1 - x^2)^{\ell/2} \frac{(2\ell)!}{2\ell!}.
\]

So

\[
Y_\ell^\ell(\theta, \phi) = (-1)^\ell \sqrt{\frac{(2\ell + 1)}{4\pi}} \frac{1}{(2\ell)!} e^{i\ell \phi} \sin^\ell \theta \frac{(2\ell)!}{2\ell!} = \sqrt{\frac{1}{\ell!}} \sqrt{\frac{(2\ell + 1)}{4\pi}} \left( -\frac{1}{2} e^{i\phi} \sin \theta \right)^\ell.
\]

To evaluate \( Y_\ell^\ell \) as a solution to the differential equation given, we first evaluate \( \partial Y / \partial \theta \):

\[
\frac{\partial Y_\ell^\ell}{\partial \theta} = \frac{\ell \cos \theta}{\sin \theta} Y_\ell^\ell.
\]

Thus we can evaluate

\[
\sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y_\ell^\ell}{\partial \theta} \right) = \ell \sin \theta \frac{\partial}{\partial \theta} \left( \cos \theta Y_\ell^\ell \right)
\]

\[
= \ell \sin \theta \left( -\sin \theta Y_\ell^\ell + \frac{\ell \cos^2 \theta}{\sin \theta} Y_\ell^\ell \right)
\]

\[
= Y_\ell^\ell \left( \ell^2 \cos^2 \theta - \ell \sin^2 \theta \right).
\]

The second derivative with respect to \( \phi \) is:

\[
\frac{\partial Y_\ell^\ell}{\partial \phi} = -\ell^2 Y_\ell^\ell,
\]

so we arrive at

\[
\sin \theta \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial Y_\ell^\ell}{\partial \theta} \right) + \frac{\partial Y_\ell^\ell}{\partial \phi} = (\ell^2 \cos^2 \theta - \ell \sin^2 \theta - \ell^2) Y_\ell^\ell
\]

\[
= -\ell(\ell + 1) \sin^2 \theta Y_\ell^\ell,
\]

which demonstrates that \( Y_\ell^\ell \) satisfies the given differential equation.

c) To show that the Legendre polynomials are orthogonal we evaluate

\[
\int_{-1}^{1} P_\ell(x) P_{\ell'}(x) dx = A(\ell) A(\ell') \int_{-1}^{1} \left( \frac{d}{dx} \right)^\ell (x^2 - 1)^{\ell'} \left( \frac{d}{dx} \right)^{\ell'} (x^2 - 1)^{\ell} dx,
\]

where we let \( A(\ell) \equiv (2\ell)!^{-1} \). Integrating by parts follows the form,

\[
\int \frac{du}{dx} v dx = uv - \int u \frac{dv}{dx} dx,
\]
so applying this once gives

\[
\frac{1}{A(\ell)A(\ell') \int_{-1}^{1} P_\ell(x) P_{\ell'}(x) \, dx} = \left( \frac{d}{dx} \right)^{\ell-1} (x^2 - 1)^\ell \left( \frac{d}{dx} \right)^{\ell'} (x^2 - 1)^{\ell'} \bigg|_{-1}^{1} \\
- \int_{-1}^{1} \left( \frac{d}{dx} \right)^{\ell-1} (x^2 - 1)^\ell \left( \frac{d}{dx} \right)^{\ell'+1} (x^2 - 1)^{\ell'} \, dx.
\]

Applying this a second time gives

\[
\frac{1}{A(\ell)A(\ell') \int_{-1}^{1} P_\ell(x) P_{\ell'}(x) \, dx} = \left( \frac{d}{dx} \right)^{\ell-1} (x^2 - 1)^\ell \left( \frac{d}{dx} \right)^{\ell'} (x^2 - 1)^{\ell'} \bigg|_{-1}^{1} \\
- \left( \frac{d}{dx} \right)^{\ell-2} (x^2 - 1)^\ell \left( \frac{d}{dx} \right)^{\ell'+1} (x^2 - 1)^{\ell'} \bigg|_{1}^{1} \\
+ \int_{-1}^{1} \left( \frac{d}{dx} \right)^{\ell-2} (x^2 - 1)^\ell \left( \frac{d}{dx} \right)^{\ell'+2} (x^2 - 1)^{\ell'} \, dx.
\]

This can continue until we run out of \( \ell \):

\[
\frac{1}{A(\ell)A(\ell') \int_{-1}^{1} P_\ell(x) P_{\ell'}(x) \, dx} = \sum_{n=1}^{n \leq \ell} (-1)^{n-1} \left( \frac{d}{dx} \right)^{\ell-n} (x^2 - 1)^\ell \left( \frac{d}{dx} \right)^{\ell'-n+1} (x^2 - 1)^{\ell'} \bigg|_{1}^{1} \\
+ (-1)^{\ell} \int_{-1}^{1} (x^2 - 1)^\ell \left( \frac{d}{dx} \right)^{\ell'+\ell} (x^2 - 1)^{\ell'} \, dx.
\]

Evaluating the first derivative term of the boundary condition, we see that it is

\[
\left( \frac{d}{dx} \right)^{\ell-n} (x^2 - 1)^\ell \bigg|_{-1}^{1} = 0
\]

as long as \( n > 0 \), since there will always be at least one overall factor of \((x^2 - 1)\) left over after taking derivatives. Therefore the boundary terms vanish. The integral also vanishes unless \( \ell = \ell' \). First consider the case when \( \ell \neq \ell' \). Without loss of generality, let \( \ell > \ell' \) (we could have always integrated by parts by pealing off derivatives from the \( \ell' \) terms instead). In that case, the expression

\[
\left( \frac{d}{dx} \right)^{\ell'+\ell} (x^2 - 1)^{\ell'}
\]

involves taking \( \ell + \ell' > 2\ell' \) derivatives of a \( 2\ell' \)-order polynomial, which is equal to 0. Only when \( \ell = \ell' \) does this not vanish. In that case

\[
\left( \frac{d}{dx} \right)^{2\ell} (x^2 - 1)^\ell = \left( \frac{d}{dx} \right)^{2\ell} (x^{2\ell}) = (2\ell)!,
\]
and therefore,
\[ \int_{-1}^{1} P_\ell(x)P_\ell'(x)dx = \delta_{\ell\ell'} \frac{(2\ell)!}{(2\ell+1)!} \int_{-1}^{1} (1 - x^2)^\ell dx. \]

We evaluate the integral by a change of variable: \( x = \cos \theta \) and \( dx = -\sin \theta d\theta \), so
\[ \int_{-1}^{1} (1 - x^2)^\ell dx = \int_{\pi}^{0} (1 - \cos^2 \theta)^\ell (-\sin \theta)d\theta = \int_{0}^{\pi} (\sin \theta)^{2\ell+1}d\theta. \]

We can evaluate this integral with integration by parts (for \( n > 1 \)):
\[ \int_{0}^{\pi} \sin^n \theta d\theta = -\cos \theta \sin^{n-1} \theta \bigg|_{0}^{\pi} + (n-1) \int_{0}^{\pi} \cos^2 \theta \sin^{n-2} \theta d\theta = (n-1) \int_{0}^{\pi} (\sin^{n-2} \theta - \sin^n \theta)d\theta \]
where \( u = \sin^{n-1} \theta \), \( du = (n-1)\sin^{n-2} \theta \cos \theta \), \( dv = \sin \theta d\theta \), \( v = -\cos \theta \). Bringing the \( \sin^n \theta \) terms together, we get
\[ \int_{0}^{\pi} \sin^n \theta d\theta = \frac{n-1}{n} \int_{0}^{\pi} \sin^{n-2} \theta d\theta. \]

Continuing the expression we get
\[ \int_{0}^{\pi} \sin^n \theta d\theta = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots 2 \times \frac{2}{3} \int_{0}^{\pi} \sin \theta d\theta = 2 \times \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots 2 \times 3 \frac{2}{3} \]
when \( n \) is odd (which is the case we care about, since we will evaluate \( n = 2\ell + 1 \)). Notice that the integral of \( \sin \theta \) evaluates to 2. Using this result gives
\[ \int_{0}^{\pi} (\sin \theta)^{2\ell+1}d\theta = 2 \times \frac{2 \cdot 4 \cdot 6 \cdots 2\ell}{3 \cdot 5 \cdot 7 \cdots 2\ell + 1}. \]

Considering the numerator separately, we have
\[ 2 \cdot 4 \cdot 6 \cdots 2\ell = 2^\ell \cdot 1 \cdot 2 \cdot 3 \cdots \ell = 2^\ell \ell!. \]

Similarly, the denominator is
\[ 1 \cdot 3 \cdot 5 \cdot 7 \cdots 2\ell + 1 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots 2\ell + 1}{2\ell!} = \frac{(2\ell + 1)!}{2\ell!} \]
where we have used the result from the numerator for the product of even numbers. Therefore
\[ \int_{0}^{\pi} (\sin \theta)^{2\ell+1}d\theta = 2 \cdot \left(\frac{2^\ell \ell!}{2\ell!}\right)^2 \frac{(2^\ell \ell!)^2}{(2\ell + 1)!}, \]
so that
\[ \int_{-1}^{1} P_\ell(x)P_\ell'(x)dx = \delta_{\ell\ell'} \frac{(2\ell)!}{(2\ell+1)!^2} \frac{(2^\ell \ell!)^2}{(2\ell + 1)!} = \left(\frac{2}{2\ell + 1}\right) \delta_{\ell\ell'}. \]
3. Question 3

a) 

\[ L_0(x) = e^x \left( \frac{d}{dx} \right)^0 \left( e^{-x} x^0 \right) = e^x e^{-x} = 1. \]

\[ L_1(x) = e^x \left( \frac{d}{dx} \right)^1 \left( e^{-x} x^1 \right) = e^x (-e^{-x} x + e^{-x}) = 1 - x. \]

\[ L_2(x) = e^x \left( \frac{d}{dx} \right)^2 \left( e^{-x} x^2 \right) = e^x \frac{d}{dx} \left( -e^{-x} x^2 + 2xe^{-x} \right) = e^x \left( e^{-x} x^2 - 2xe^{-x} + 2e^{-x} - 2xe^{-x} \right) = x^2 - 4x + 2. \]

\[ L_3(x) = e^x \left( \frac{d}{dx} \right)^3 \left( e^{-x} x^3 \right) = e^x \left( \frac{d}{dx} \right)^2 \left( -e^{-x} x^3 + 3x^2 e^{-x} \right) = e^x \left( e^{-x} x^3 - 3x^2 e^{-x} + 6xe^{-x} - 3x^2 e^{-x} \right) = e^x \left( -e^{-x} x^3 + 3x^2 e^{-x} - 6xe^{-x} + 3x^2 e^{-x} - 6e^{-x} - 6xe^{-x} + 3x^2 e^{-x} \right) = -x^3 + 9x^2 - 18x + 6. \]

b) The ground state wavefunction is

\[ \psi_{100} = \sqrt{\left( \frac{2}{a} \right)^3 \frac{1}{2} e^{-r/a}} \frac{1}{2\sqrt{\pi}} = \frac{1}{\sqrt{\pi}a^3} e^{-r/a}. \]

We can compute the expectation value of \( r \) according to

\[ \langle r \rangle = \int_0^\infty \int_0^\pi \int_0^{2\pi} r \left( \frac{1}{\sqrt{\pi}a^3} e^{-r/a} \right)^2 r^2 dr d\theta d\phi \]

\[ = \frac{4\pi}{\pi a^3} \int_0^\infty r^3 e^{-2r/a} dr \]

\[ = \frac{4}{a^3} \left[ -\frac{a}{2} r^3 e^{-2r/a} \right]_0^\infty + \int_0^\infty \frac{3a}{2} r^2 e^{-2r/a} dr \]

\[ = \frac{6}{a^2} \left[ -\frac{a}{2} r^2 e^{-2r/a} \right]_0^\infty + \int_0^\infty a e^{-2r/a} dr \]

\[ = \frac{6}{a} \left[ -\frac{a}{2} e^{-2r/a} \right]_0^\infty + \int_0^\infty \frac{a}{2} e^{-2r/a} dr \]

\[ = -\frac{3a}{2} e^{-2r/a} \bigg|_0^\infty = \frac{3}{2}. \]
Similarly,
\[
\langle r^2 \rangle = \frac{4}{a^3} \int_0^\infty r^4 e^{-2r/a} dr
\]
\[
= \frac{4}{a^3} \left[ -\frac{a}{2} r^4 e^{-2r/a} \bigg|_0^\infty + \int_0^\infty \frac{4a}{2} r^3 e^{-2r/a} dr \right]
\]
\[
= \frac{8}{a^2} \left[ -\frac{a}{2} r^3 e^{-2r/a} \bigg|_0^\infty + \int_0^\infty \frac{3a}{2} r^2 e^{-2r/a} dr \right]
\]
\[
= \frac{12}{a} \left[ -\frac{a}{2} r^2 e^{-2r/a} \bigg|_0^\infty + \int_0^\infty \frac{a}{2} r e^{-2r/a} dr \right]
\]
\[
= \frac{12}{a} \left[ -\frac{a}{2} r e^{-2r/a} \bigg|_0^\infty + \int_0^\infty \frac{a}{2} e^{-2r/a} dr \right]
\]
\[
= -3a^2 e^{-2r/a} \bigg|_0^\infty = 3a^2
\]

The variance is
\[
\sigma^2 = \langle r^2 \rangle - \langle r \rangle^2 = 3a^2 - \frac{9}{4} a^2 = \frac{3}{4} a^2,
\]
so
\[
\sigma_r = \frac{\sqrt{3}}{2} a.
\]

c) The probability that the ground state would be found within a radius of size \(b\) from the center is
\[
P = \frac{4}{a^3} \int_0^b r^2 e^{-2r/a} dr
\]
\[
= \frac{4}{a^3} \left[ -\frac{a}{2} r^2 e^{-2r/a} \bigg|_0^b + \int_0^b a r e^{-2r/a} dr \right]
\]
\[
= \frac{4}{a^3} \left[ -\frac{ab^2}{2} e^{-2b/a} - \frac{a^2}{2} r e^{-2r/a} \bigg|_0^b + \int_0^b \frac{a^2}{2} e^{-2r/a} dr \right]
\]
\[
= \frac{4}{a^3} \left[ -\frac{ab^2}{2} e^{-2b/a} - \frac{a^2 b}{2} e^{-2b/a} - \frac{a^3}{4} e^{-2r/a} \bigg|_0^b \right]
\]
\[
= \frac{4}{a^3} \left[ \frac{a^3}{4} - e^{-2b/a} \left( \frac{ab^2}{2} + \frac{a^2 b}{2} + \frac{a^3}{4} \right) \right]
\]
\[
= 1 - e^{-2b/a} \left( \frac{2b^2}{a^2} + \frac{b}{a} + 1 \right).
\]
d) Expanding around $\epsilon \equiv 2b/a$, we get

\[
P = 1 - e^{-\epsilon} \left( \frac{1}{2} \epsilon^2 + \epsilon + 1 \right)
\]

\[
= 1 - \left( 1 - \epsilon + \frac{\epsilon^2}{2} - \frac{\epsilon^3}{6} + \cdots \right) \left( \frac{1}{2} \epsilon^2 + \epsilon + 1 \right)
\]

\[
= 1 - \left( \frac{\epsilon^2}{2} + \epsilon + 1 - \frac{\epsilon^3}{2} - \epsilon^2 - \epsilon + \frac{\epsilon^3}{2} + \frac{\epsilon^2}{2} - \frac{\epsilon^3}{6} + O(\epsilon^4) \right)
\]

\[
\approx \frac{\epsilon^3}{6} = \frac{4}{3} \left( \frac{b}{a} \right)^3.
\]

Plugging in $b = 10^{-15}$ m and $a = 0.5 \times 10^{-10}$ m, we get $b/a = 2 \times 10^{-5}$, so $P \approx 1.1 \times 10^{-14}$. 
