(1) The time-independent solutions to the infinite square well are
\[ \psi_n(x) = \sqrt{\frac{2}{a}} \sin \left( \frac{n \pi}{a} x \right), \]
for positive integers \( n \) and length \( a \), and have energies
\[ E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}. \]

(a) Prove that the solutions are mutually orthogonal, that is
\[ \int \psi_m(x)^* \psi_n(x) \, dx = \delta_{mn}, \]
where \( \delta_{mn} \) is the Kronecker delta, defined as 0 when \( m \neq n \) and 1 when \( m = n \).

(b) Calculate \( \langle x \rangle \), \( \langle p \rangle \), \( \langle x^2 \rangle \), \( \langle p^2 \rangle \), \( \sigma_x \) and \( \sigma_p \) for the \( n \)th stationary state of the infinite square well. Check that the uncertainty principle is satisfied. Which state comes closest to the uncertainty limit?

(2) A particle in the infinite square well has as its initial wave function an even mixture of the first two stationary states:
\[ \Psi(x, 0) = A[\psi_1(x) + \psi_2(x)], \]
where \( \psi_1 \) and \( \psi_2 \) are defined above.

(a) Normalize \( \Psi(x, 0) \). Recall (from question 3 of homework 5) that once you have normalized \( \Psi \) at a fixed time it stays normalized. Hint: you will want to use the orthogonality condition you derived above.

(b) Find \( \Psi(x, t) \) and \( |\Psi(x, t)|^2 \), using \( \omega \equiv \pi^2 \hbar / 2ma^2 \). Explicitly show that \( \Psi(x, t) \) satisfies the Schrödinger equation. Express \( |\Psi(x, t)|^2 \) in terms of sinusoidal functions of time, eliminating the exponentials with the help of Euler’s formula: \( e^{i\theta} = \cos \theta + i \sin \theta \).

(c) Calculate \( \langle x \rangle \). Note that it oscillates with time. What is the frequency of oscillation? What is the amplitude of the oscillation?

(d) Calculate \( \langle p \rangle \) Hint: we showed in class how \( \langle x \rangle \) relates to \( \langle p \rangle \).

(e) Calculate the expectation value of the Hamiltonian operator \( H \equiv \frac{p^2}{2m} + V \) for \( \Psi(x, t) \). How does it compare with the energies of the first two stationary states \( E_1 \) and \( E_2 \)? Hint: don’t forget to use orthogonality (Question 1a).

(f) A classical particle in this well would bounce back and forth between the walls. If its energy is equal to the expectation value you just calculated, what is the frequency of the classical motion? How does it compare with the quantum frequency in part (c)?
(3) The time independent Schrödinger equation for the 1D simple harmonic oscillator can be re-expressed as

\[(a_+ a_- + \frac{1}{2} \hbar \omega) \psi_n = E_n \psi_n\]

\[(a_- a_+ - \frac{1}{2} \hbar \omega) \psi_n = E_n \psi_n\]

where \(n\) is an integer, and the raising and lowering operators are defined as

\[a_\pm \equiv \frac{1}{\sqrt{2m}} \left( \frac{\hbar}{i} \frac{d}{dx} \pm im\omega x \right)\]

The solutions are

\[\psi_n(x) = A_n (a_+)^n \exp \left( -\frac{m\omega}{2\hbar} x^2 \right)\]

and the corresponding energies for these solutions are \(E_n = (n + \frac{1}{2}) \hbar \omega\). The raising and lowering operators generate new solutions to the time independent Schrödinger equation for \(V(x) = \frac{1}{2} m\omega^2 x^2\), but these solutions are not correctly normalized (hence the normalization term \(A_n\)). While \(a_+ \psi_n\) is proportional to \(\psi_{n+1}\) and \(a_- \psi_n\) is proportional to \(\psi_{n-1}\), we still need to know the precise proportionality constants.

(a) Use integration by parts to show that

\[\int_{-\infty}^{\infty} |a_+ \psi_n|^2 dx = \int_{-\infty}^{\infty} \psi^*_n (a_- a_+ \psi_n) dx,\]

and similarly,

\[\int_{-\infty}^{\infty} |a_- \psi_n|^2 dx = \int_{-\infty}^{\infty} \psi^*_n (a_+ a_- \psi_n) dx.\]

Then use the (time independent) Schrödinger equation to show that

\[\int_{-\infty}^{\infty} |a_+ \psi_n|^2 dx = (n + 1) \hbar \omega,\]

\[\int_{-\infty}^{\infty} |a_- \psi_n|^2 dx = n \hbar \omega.\]

From this we see that if we assume \(\psi\) is normalized properly then

\[a_+ \psi_n = i \sqrt{(n + 1) \hbar \omega} \psi_{n+1},\]

\[a_- \psi_n = -i \sqrt{n \hbar \omega} \psi_{n-1},\]

where we have included \(i\)'s to keep the wavefunctions real. Now use these equations to show that

\[A_n = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} \frac{(-i)^n}{\sqrt{n!(\hbar \omega)^n}}.\]

To get this, you will need to normalize \(\psi_0\) “by hand.”

(b) compute \(\langle x \rangle\), \(\langle p \rangle\), \(\langle x^2 \rangle\), and \(\langle p^2 \rangle\) for the ground state wavefunction \(\psi_0(x)\). Check the uncertainty principle for this state.