(1) The needle on a broken speedometer is free to swing so that if you give it a flick it is equally likely to come to rest at any angle between 0 and $\pi$.

(a) What is the probability density, $\rho(\theta)$, such that $\rho(\theta)d\theta$ is the probability that the needle will come to rest between $\theta$ and $(\theta + d\theta)$? Make sure that the total probability is 1.

(b) Compute $\langle \theta \rangle$, $\langle \theta^2 \rangle$, and $\sigma$ for the probability distribution.

(c) Compute $\langle \sin \theta \rangle$, $\langle \cos \theta \rangle$, and $\langle \cos^2 \theta \rangle$.

(2) Consider the same problem as the previous, but now we are interested in the $x$-coordinate of the needle’s point, where the needle has length $r$.

(a) What is the probability density, $\rho(x)$, such that the $\rho(x)dx$ is the probability that the needle point will have an $x$ coordinate between $x$ and $x + dx$? Remember that you know the probability that the needle is in the range from $\theta$ to $\theta + d\theta$, so the question is what interval $dx$ corresponds to $d\theta$.

(b) Compute $\langle x \rangle$, $\langle x^2 \rangle$, and $\sigma$ for the probability distribution.

(3) Show that

$$\frac{d}{dt} \int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 0.$$  

This demonstrates that once the wavefunction is normalized, it stays normalized, regardless of how it evolves with time. Hint: follow the approach of the in-class derivation of $\langle p \rangle = m\langle x \rangle/dt$.

(4) Suppose the following wave function is a solution to the Schrödinger equation:

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t},$$

where $A$, $\lambda$, and $\omega$ are positive real constants.

(a) Find the value of $A$.

(b) Determine the expectation values of $x$, $x^2$, and $\sigma$.

(c) What is the probability that the particle will be found between $x = [-\sigma, \sigma]$?

(5) Consider the wave function of a particle with mass $m$

$$\Psi(x,t) = Ae^{-a[(mx^2/\hbar)+it]},$$

where $A$ and $a$ are positive real constants.

(a) Find $A$. Hint: $\int_{-\infty}^{\infty} e^{-a(x+b)^2} dx = \sqrt{\frac{\pi}{a}}$.
(b) For what potential energy function $V(x)$ does $\Psi$ satisfy the Schrödinger equation?
(c) Calculate the expectation values of $x$, $x^2$, $p$, and $p^2$.
(d) Find $\sigma_x$ and $\sigma_p$. Does this satisfy the uncertainty principle?

(6) Suppose that a solution to the Schrödinger equation gives rise to three wavefunctions with the given domains:

\[ \Psi_1(x) = A e^{kx} \quad (\infty \leq x < 0) \]
\[ \Psi_2(x) = Bx^2 + Cx + D \quad (0 \leq x < L) \]
\[ \Psi_3(x) = 0 \quad (x \geq L). \]

(a) What values must $B$, $C$, and $D$ take in terms of $A$ in order for these solutions to satisfy the continuity conditions, namely that $\Psi$ and $d\Psi/dx$ are continuous? Note that the solution $\Psi_3(x) = 0$ requires an infinitely discontinuous potential at $x = L$, so the continuity condition for $d\Psi/dx$ does not apply at $x = L$.
(b) Derive an expression that $A$, $B$, $C$, $D$, $k$, and $L$ must satisfy in order for the wavefunction to be normalized properly.