1. Question 1

In the frame of the cop car the outlaws are traveling at

\[ u' = \frac{u - v}{1 - uv/c^2} = \frac{3/4 - 1/2}{1 - 3/4 \cdot 1/2} c = \frac{2}{5} c, \]

which is faster than \( \frac{1}{5} c \), so the bullet does not reach the getaway car.

2. Question 2

Since the particle masses are equal, the center of momentum (c.o.m.) frame is achieved when the velocities of the two particles are equal and opposite. Let’s call the magnitude of the velocity of the two particles in the c.o.m. frame \( v' \) and c.o.m. frame velocity (with respect to the lab) \( V \). The moving particle has its velocity transformed according to the equation

\[ v' = \frac{v - V}{1 - vV/c^2}, \]

and the stationary particle has its velocity transformed according to the equation

\[ -v' = \frac{0 - V}{1 - 0 \cdot V/c^2} = -V. \]

Therefore, we want to solve the following equation for \( V \):

\[ V = \frac{v - V}{1 - vV/c^2}, \]

Multiplying the denominator by both sides and re-arranging terms gives

\[ \frac{v}{c^2} V^2 - 2V + v = 0, \]

which is solved by the quadratic equation:

\[ V = \frac{2 \pm \sqrt{4 - 4v^2/c^2}}{2v/c^2} = \frac{1 \pm \sqrt{1 - \beta^2}}{\beta} c \]

In the limit where \( \beta \to 0 \), the positive solution is divergent, so we pick the negative solution:

\[ V = \frac{1 - \sqrt{1 - \beta^2}}{\beta} c. \]
If we Taylor expand the radical, we recover the classical solution:

\[ V = \frac{c}{\beta}(1 - 1 + \beta^2/2 + \cdots) \approx \frac{v}{2}. \]

3. Question 3

Conservation of energy requires that

\[ \epsilon_i + m_0 c^2 = \epsilon_f + \gamma m_0 c^2 = \epsilon_f + \sqrt{(\gamma m_0 u c)^2 + m_0^2 c^4} \]

Conservation of momentum along the direction of the \( x \)-axis requires that

\[ \frac{\epsilon_i}{c} + \frac{\epsilon_f}{c} \cos \theta = \gamma m_0 u \cos \phi \]

and conservation of momentum along the direction \( y \)-axis requires that

\[ \frac{\epsilon_f}{c} \sin \theta = \gamma m_0 u \sin \phi. \]

Squaring and summing the conservation of momentum equations gives

\[ (\gamma m_0 u)^2 (\sin^2 \phi + \cos^2 \phi) = (\gamma m_0 u)^2 = \frac{\epsilon_i^2}{c^2} + \frac{\epsilon_f^2}{c^2} + 2 \frac{\epsilon_i \epsilon_f}{c^2} \cos \theta, \]

which we can plug into the conservation of energy equation above to get:

\[ \epsilon_i + m_0 c^2 = \epsilon_f + \sqrt{\frac{\epsilon_i^2}{c^2} + \frac{\epsilon_f^2}{c^2} + 2 \frac{\epsilon_i \epsilon_f}{c^2} \cos \theta + m_0^2 c^4}. \]

Isolating the radical and squaring gives

\[ (\epsilon_i + m_0 c^2 - \epsilon_f)^2 = \epsilon_i^2 + m_0^2 c^4 + \epsilon_f^2 + 2 \epsilon_i m_0 c^2 - 2 \epsilon_i \epsilon_f - 2 \epsilon_f m_0 c^2 = \epsilon_i^2 + \epsilon_f^2 + 2 \epsilon_i \epsilon_f \cos \theta + m_0^2 c^4, \]

and solving for \( \epsilon_f \) yields

\[ \epsilon_f = \frac{m_0 c^2}{1 + \cos \theta + m_0 c^2 / \epsilon_i}, \]

which is now conveniently expressed in terms of only known variables. To get the angle \( \phi \), we take the ratio of the conservation of momentum equations to give

\[ \cot \phi = \frac{\epsilon_i / \epsilon_f + \cos \theta}{\sin \theta}. \]
Plugging in our expression for $\epsilon_f$ yields
\[
\cot \phi = \frac{\epsilon_i (1 + \cos \theta) + m_0 c^2}{m_0 c^2 \sin \theta} + \cos \theta
\]
\[
= \frac{\epsilon_i (1 + \cos \theta) + m_0 c^2 (1 + \cos \theta)}{m_0 c^2 \sin \theta}
\]
\[
= \left(1 + \frac{\epsilon_i}{m_0 c^2}\right) \left(\frac{1 + \cos \theta}{\sin \theta}\right)
\]
\[
= \left(1 + \frac{\epsilon_i}{m_0 c^2}\right) \tan \frac{\theta}{2}.
\]

4. Question 4

Conservation of energy requires that
\[
E_0 + E_i = E_f + \sqrt{p_f^2 c^2 + m_0^2 c^4}
\]
where $E_i$ is the initial energy of the electron, $p_f$ is the magnitude of the momentum of the electron after scattering, and $E_0$ and $E_f$ are the initial and final energies of the photon, respectively. Conservation of momentum along the direction of the $x$-axis requires that
\[
E_0/c - p_i = p_f \cos \theta,
\]
where $p_i$ is the initial momentum of the electron, and conservation of momentum along the direction $y$-axis requires that
\[
E_f/c = p_f \sin \theta.
\]

Squaring and summing the conservation of momentum equations gives
\[
p_f^2 c^2 = E_0^2 + p_i^2 c^2 - 2E_0p_i c + E_f^2,
\]
which we can plug in to the conservation of energy equation, yielding
\[
E_0 + E_i = E_f + \sqrt{E_0^2 + p_i^2 c^2 - 2E_0p_i c + E_f^2 + m_0^2 c^4} = E_f + \sqrt{E_0^2 - 2E_0p_i c + E_f^2 + E_i^2}.
\]

Isolating the radical and squaring both sides gives
\[
(E_0 + E_i - E_f)^2 = E_0^2 - 2E_0p_i c + E_f^2 + E_i^2,
\]
which after expanding and cancelling terms simplifies to
\[
(E_0 - E_f)E_i - E_0E_f = -E_0p_i c.
\]
Solving for $E_f$ yields
\[
E_f = \frac{E_0 E_i + E_0p_i c}{E_i + E_0} = \frac{E_0 (1 + p_i c/E_i)}{1 + E_0/E_i} = \frac{E_0 (1 + \beta)}{1 + E_0/E_i}.
\]
5. Question 5

Assuming perfect reflection and that the sphere is large compared to the spot of the laser light, the change in momentum per photon is

$$\Delta p = \frac{2E}{c},$$

where $E$ is the energy of the photon. The force is

$$F = \frac{\Delta p}{\Delta t} = \frac{2E}{c \Delta t} = \frac{2P}{c},$$

where $P$ is the power (work per unit time). For levitation in equilibrium,

$$F = mg = \frac{4}{3} \pi r^3 \rho g.$$

Equating the two forces gives

$$\frac{4}{3} \pi r^3 \rho g = \frac{2P}{c},$$

and solving for $r$ gives

$$r = \left( \frac{3P}{2 \pi \rho gc} \right)^{\frac{1}{3}} = \left( \frac{3 \cdot 1 \times 10^3 \text{ W}}{2\pi \cdot (2.7 \times 10^3 \text{ kg/m}^3) \cdot (9.8 \text{ m/s}^2) \cdot (3 \times 10^8 \text{ m/s})} \right)^{\frac{1}{3}} \approx 0.39 \text{ mm}$$

6. Question 6

Working in A’s frame, the decay must be back-to-back, so we can arbitrarily choose the axis of the decay. Energy conservation requires

$$m_A c^2 = E_B + E_C = E_B + \sqrt{p_C^2 c^2 + m_C^4},$$

where I use the notation $p \equiv |\mathbf{p}|$. Momentum conservation requires that $p_B^2 = p_C^2$, so

$$m_A c^2 = E_B + \sqrt{p_B^2 c^2 + m_C^4} = E_B + \sqrt{E_B^2 - m_B^2 c^4 + m_C^4}.$$

Rearranging terms of squaring both sides gives

$$(m_A c^2 - E_B)^2 = E_B^2 - m_B^2 c^4 + m_C^4$$

$$m_A^2 c^4 - 2m_A c^2 E_B + E_B^2 = E_B^2 - m_B^2 c^4 + m_C^4$$

$$m_A^2 c^4 - 2m_A c^2 E_B = -m_B^2 c^4 + m_C^4$$

which finally gives

$$E_B = \frac{m_A^2 + m_B^2 - m_C^2}{2m_A} c^2.$$

By symmetry,

$$E_C = \frac{m_A^2 - m_B^2 + m_C^2}{2m_A} c^2.$$
To determine the momentum, we can use $p_B^2 = E_B^2/c^2 - m_B^2 c^2$. By using the answer to the first part, we get

\[
p_B^2 = \frac{(m_A^2 + m_B^2 - m_C^2)^2}{4m_A^2} c^2 - m_B^2 c^2
\]

\[
= \frac{m_A^4 + m_B^4 + m_C^4 + 2m_A^2m_B^2 - 2m_A^2m_C^2 - 2m_B^2m_C^2 - 4m_A^2m_B^2c^2}{4m_A^2}
\]

\[
= \frac{\lambda(m_A^2, m_B^2, m_C^2)}{4m_A^2} c^2.
\]

Taking the square root of both sides gives the result

\[
p_B = p_C = \frac{\sqrt{\lambda(m_A^2, m_B^2, m_C^2)}}{2m_A} c.
\]

7. Question 7

Conservation of energy requires that

\[
E_\pi = E_\mu + E_\nu = E_\mu + p_\nu c.
\]

Conservation of momentum along the direction of the pion’s momentum requires that

\[
p_\pi = p_\mu \cos \theta,
\]

and conservation of momentum along the direction orthogonal to the pion’s momentum requires that

\[
p_\nu = p_\mu \sin \theta.
\]

Squaring and summing the conservation of momentum equations gives

\[
p_\mu^2 \cos^2 \theta + p_\mu^2 \sin^2 \theta = p_\mu^2 = p_\pi^2 + p_\nu^2.
\]

Using the energy-momentum relation, we can rewrite the conservation of energy equation as

\[
E_\pi = \sqrt{p_\pi^2 c^2 + m_\mu^2 c^4 + p_\nu c} = \sqrt{p_\pi^2 c^2 + p_\nu^2 c^2 + m_\mu^2 c^4 + p_\nu c}.
\]

Re-arranging terms and squaring gives

\[
(E_\pi - p_\nu c)^2 = p_\pi^2 c^2 + p_\nu^2 c^2 + m_\mu^2 c^4,
\]

from which we can solve for $p_\nu c$, yielding

\[
p_\nu c = \frac{E_\pi^2 - p_\pi^2 c^2 - m_\mu^2 c^4}{2E_\pi} = \frac{m_\mu^2 - m_\pi^2 c^2}{2\gamma m_\pi} c^2.
\]
We can solve for \( \tan \theta \) by re-arranging the momentum conservation equations and taking the ratio

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} = \frac{p_\nu}{p_\pi} = \frac{m^2_\pi - m^2_\mu}{(2\gamma m_\pi)(\gamma m_\pi v)} c = \frac{1 - m^2_\mu/m^2_\pi}{2\beta^2 \gamma^2}.
\]