1. Question 1

a) We have from the Bethe–Weizsacker equation,
\[ B = a_V A - a_A A^{2/3} - 0.72Z^2 A^{-1/3} - a_S (A - 2Z)^2 A^{-1} + \delta. \]
Maximizing the stability for a fixed \( A \) requires taking the derivative of \( B \) with respect to \( Z \):
\[ \frac{\partial B}{\partial Z} = -1.44ZA^{-1/3} + 4a_S(A - 2Z)A^{-1} = 0. \]

Note that the second derivative with respect to \( Z \) is always negative, implying a maximum for all values of \( A \), which is consistent with what we want. We then solve for \( Z \):
\[ 0 = -Z(1.44A^{-1/3} + 8a_SA^{-1}) + 4a_S, \]
yielding
\[ Z = \frac{a_SA}{0.36A^{2/3} + 2a_S}. \]

b) When a nucleus undergoes beta decay, the value of \( A \) is unchanged but the value of \( Z \) changes by one unit. If \( B \) is maximized with respect to a change in \( Z \), then the daughter nucleus would have less binding energy than the parent, which is kinematically forbidden.

c) For \( A = 6 \), the above formula gives \( Z \approx 3 \), which corresponds to \( ^6\text{Li} \), a stable element. For \( A = 28 \), \( Z \approx 13 \), i.e. \( ^{28}\text{Al} \), which not \( \beta \)-stable. For \( A = 44 \), \( Z \approx 20 \), which corresponds to the stable element \( ^{44}\text{Ca} \).

2. Question 2

a) This can be computed non-relativistically. The Coulomb potential between two protons is
\[ V(r) = \frac{1}{4\pi\epsilon_0} \frac{e^2}{r} \approx 9.0 \times 10^9 \text{ N} \cdot \text{m}^2 \cdot \text{C}^{-2} \frac{(1.6 \times 10^{-19} \text{ C})^2}{10^{-15} \text{ m}} \approx 2.3 \times 10^{-13} \text{ J} \approx 1.4 \text{ MeV}. \]
b) The temperature is

\[ T = \frac{2K}{3k} \approx \frac{2}{3} \left( \frac{1.4 \times 10^6 \text{eV}}{3.6 \times 10^{-5} \text{eV} \cdot \text{K}^{-1}} \right) \approx 1.1 \times 10^{10} \text{K} \]

\[
\begin{align*}
    T & = \frac{2K}{3k} \\
    & \approx \frac{2}{3} \left( \frac{1.4 \times 10^6 \text{eV}}{3.6 \times 10^{-5} \text{eV} \cdot \text{K}^{-1}} \right) \\
    & \approx 1.1 \times 10^{10} \text{K}
\end{align*}
\]

The reason that fusion proceeds in the sun anyway is because temperature corresponds only to a mean kinetic energy, so some of the protons nevertheless have a high enough kinetic energy to overcome the Coulomb barrier.

3. Question 3

The four momentum of the initial system, evaluated in the lab frame, is

\[ \mathbf{p} = (p_x, 0, 0, iE_x/c + im_xc), \]

where we have chosen the $x$-axis arbitrarily to be the axis of the collision. The four momentum of the final system, evaluated in the center of mass frame, is

\[
\mathbf{k} = \left( 0, 0, 0, \frac{i}{c} \sum_j E_j \right),
\]

where the sum is over the energies of all the final state particles. However, since we are interested in the threshold energy, \( \sum_j E_j = \mu c^2 \), where \( \mu \equiv \sum_m m_j \), since at threshold all of the final state particles are produced at rest. Since four momentum is conserved and the norm is invariant, we equate \( \mathbf{p}^2 \) and \( \mathbf{k}^2 \), giving

\[
p_x^2 c^2 - (E_x + m_x c^2)^2 = -\mu^2 c^4.
\]

Substituting \( E_x^2 = p_x^2 c^2 + m_x^2 c^4 \) we have

\[
\begin{align*}
    \mu^2 c^4 &= -p_x^2 c^2 + E_x^2 + m_x^2 c^4 + 2E_x m_x c^2 \\
    &= m_x^2 c^4 + m_x^2 c^4 + 2E_x m_x c^2,
\end{align*}
\]

so that re-arranging terms gives

\[ E_x = \frac{\mu^2 - m_x^2 - m_x^2 c^2}{2m_x} \]
The kinetic energy of the colliding particle is \( K = E_x - m_x c^2 \), so

\[
K_{th} = \frac{\mu^2 - m_x^2 - m_X^2 c^2 - m_x c^2}{2m_X} \\
= \frac{\mu^2 - m_x^2 - m_X^2 - 2m_X m_x c^2}{2m_X} \\
= \frac{\mu^2 - (m_x + m_X)^2 c^2}{2m_X} \\
= \frac{(\mu + [m_x + m_X])(\mu - [m_x + m_X])}{2m_X} c^2.
\]

Because \( Q/c^2 \) is the difference in total mass between the initial and final state, \( Q = (m_x + m_X - \mu)c^2 \), so

\[
K_{th} = -Q \frac{\mu + m_x + m_X}{2m_X},
\]

or equivalently,

\[
K_{Th} = -Q \cdot \frac{\text{Sum of all masses}}{2 \times \text{target mass}}.
\]