This exam is (strictly) 180 minutes long. You may avail yourself of a calculator (no computers) and your notes (no textbook). Please write your answers in the exam. Multiple choice questions are worth one point each and have only one correct answer. Blue books are provided for you to write your answers to the last two (bonus) questions (worth three points each). In case there are any inconsistencies in numerical values of constants between what is shown below and what is in the notes, use the numbers shown below.

Name:

Student ID:
Elementary charge $e = 1.6 \times 10^{-19}$ C
1 electron volt (eV) = $1.6 \times 10^{-19}$ J
Speed of light $c = 3 \times 10^8$ m/s
Planck’s constant $h = 6.63 \times 10^{-34}$ J s = 1240 nm eV/c
$h = h/2\pi$
Avogadro’s number = $6.02 \times 10^{23}$ molecules/mole
Electron mass = $9.11 \times 10^{-31}$ kg = 0.511 MeV/c²
Proton mass = $1.673 \times 10^{-27}$ kg = 938.3 MeV/c²
Neutron mass = $1.675 \times 10^{-27}$ kg = 939.6 MeV/c²
Fine structure constant $\alpha = 1/137$
Hubble constant $H = 0.0215$ m/s/light year = 70 km/s/Mpc
1 parsec (pc) = 3.26 light years

Powers of ten:

<table>
<thead>
<tr>
<th>femto(f)</th>
<th>pico(p)</th>
<th>nano(n)</th>
<th>micro(µ)</th>
<th>milli(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-15}$</td>
<td>$10^{-12}$</td>
<td>$10^{-9}$</td>
<td>$10^{-6}$</td>
<td>$10^{-3}$</td>
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</tbody>
</table>

<p>| THE LEPTONS (all spin-$\frac{1}{2}$) |
|-------------------------------|-----|-----|-----|-----|-----|</p>
<table>
<thead>
<tr>
<th>Mass (MeV/c²)</th>
<th>Common decays</th>
<th>$L_e$</th>
<th>$L_\mu$</th>
<th>$L_\tau$</th>
<th>Antiparticle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^-$ (.511)</td>
<td>Stable</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$e^+$</td>
</tr>
<tr>
<td>$\nu_e$ (&gt; 0)</td>
<td>Stable</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$\bar{\nu}_e$</td>
</tr>
<tr>
<td>$\mu^-$ (106)</td>
<td>$e^-\bar{\nu}<em>e\nu</em>\mu$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\mu^+$</td>
</tr>
<tr>
<td>$\nu_\mu$ (&gt; 0)</td>
<td>Stable</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>$\bar{\nu}_\mu$</td>
</tr>
<tr>
<td>$\tau^-$ (1777)</td>
<td>$\pi^-\pi^0\nu_\tau$, $e^-\bar{\nu}<em>e\nu</em>\tau$, $\mu^-\bar{\nu}<em>\mu\nu</em>\tau$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\tau^+$</td>
</tr>
<tr>
<td>$\nu_\tau$ (&gt; 0)</td>
<td>Stable</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\bar{\nu}_\tau$</td>
</tr>
</tbody>
</table>

| THE QUARKS (all spin-$\frac{1}{2}$) |
|----------------------------------|-----|-----|-----|-----|-----|
| Name | Symbol | Charge | S | C | B | T | Antiparticle |
| Down     | $d$ | $-1/3$ | 0 | 0 | 0 | 0 | $\bar{d}$ |
| Up        | $u$ | $+2/3$ | 0 | 0 | 0 | 0 | $\bar{u}$ |
| Strange   | $s$ | $-1/3$ | -1 | 0 | 0 | 0 | $\bar{s}$ |
| Charm    | $c$ | $+2/3$ | 0 | +1 | 0 | 0 | $\bar{c}$ |
| Bottom   | $b$ | $-1/3$ | 0 | 0 | -1 | 0 | $\bar{b}$ |
| Top       | $t$ | $+2/3$ | 0 | 0 | 0 | +1 | $\bar{t}$ |
### HADRONS (strongly interacting particles)

Baryon number = +1 for baryons, –1 for antibaryons, 0 for all others

S = strangeness, C = charm, B = bottomness

#### SOME BARYONS (all are fermions: half-integer spin)

<table>
<thead>
<tr>
<th>Mass (MeV)</th>
<th>Common decays</th>
<th>S</th>
<th>C</th>
<th>B</th>
<th>Antiparticle</th>
</tr>
</thead>
<tbody>
<tr>
<td>p (938)</td>
<td>Stable</td>
<td>0</td>
<td>0</td>
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</tr>
<tr>
<td>n (940)</td>
<td>$pe^-\bar{\nu}_e$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>⚫</td>
</tr>
<tr>
<td>$\Lambda$ (1116)</td>
<td>$p\pi^-, n\pi^0$</td>
<td>−1</td>
<td>0</td>
<td>0</td>
<td>⚫</td>
</tr>
<tr>
<td>$\Sigma^+$ (1189)</td>
<td>$p\pi^0, n\pi^+$</td>
<td>−1</td>
<td>0</td>
<td>0</td>
<td>⚫</td>
</tr>
<tr>
<td>$\Sigma^0$ (1193)</td>
<td>$\Lambda\gamma$</td>
<td>−1</td>
<td>0</td>
<td>0</td>
<td>⚫</td>
</tr>
<tr>
<td>$\Sigma^-$ (1197)</td>
<td>$n\pi^-$</td>
<td>−1</td>
<td>0</td>
<td>0</td>
<td>⚫</td>
</tr>
<tr>
<td>$\Xi^0$ (1315)</td>
<td>$\Lambda\pi^0$</td>
<td>−2</td>
<td>0</td>
<td>0</td>
<td>⚫</td>
</tr>
<tr>
<td>$\Xi^-$ (1321)</td>
<td>$\Lambda\pi^-$</td>
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<td>0</td>
<td>0</td>
<td>⚫</td>
</tr>
<tr>
<td>$\Omega^-$ (1672)</td>
<td>$\Lambda K^-, \Xi^0\pi^-$</td>
<td>−3</td>
<td>0</td>
<td>0</td>
<td>⚫</td>
</tr>
<tr>
<td>$\Lambda_c^+$ (2285)</td>
<td>Various</td>
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<td>+1</td>
<td>0</td>
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<tr>
<td>$\Lambda_b^+$ (5624)</td>
<td>Various</td>
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<td>0</td>
<td>−1</td>
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</table>

#### SOME MESONS (all are bosons: integer spin)

<table>
<thead>
<tr>
<th>Mass (MeV)</th>
<th>Common decays</th>
<th>S</th>
<th>C</th>
<th>B</th>
<th>Antiparticle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi^+$ (140)</td>
<td>$\mu^+\nu_\mu$</td>
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<td>0</td>
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<td>⚫</td>
</tr>
<tr>
<td>$\pi^0$ (135)</td>
<td>$\gamma\gamma$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Self</td>
</tr>
<tr>
<td>$\eta^0$ (547)</td>
<td>$2\gamma, 3\pi^0,...$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Self</td>
</tr>
<tr>
<td>$K^+$ (494)</td>
<td>$\mu^+\nu_\mu, \pi^+\pi^0$</td>
<td>+1</td>
<td>0</td>
<td>0</td>
<td>⚫</td>
</tr>
<tr>
<td>$K^0$ (498)</td>
<td>$2\pi, 3\pi,...$</td>
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<td>0</td>
<td>⚫</td>
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<tr>
<td>$D^+$ (1869)</td>
<td>$K^\pm +..., K^0 +...$</td>
<td>0</td>
<td>+1</td>
<td>0</td>
<td>⚫</td>
</tr>
<tr>
<td>$D^0$ (1865)</td>
<td>$K^\pm +..., K^0 +...$</td>
<td>0</td>
<td>+1</td>
<td>0</td>
<td>⚫</td>
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<tr>
<td>$D_s^+$ (1969)</td>
<td>$K^\pm +..., K^0 +...$</td>
<td>+1</td>
<td>+1</td>
<td>0</td>
<td>⚫</td>
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<tr>
<td>$J/\psi$ (3097)</td>
<td>Various</td>
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<td>0</td>
<td>Self</td>
</tr>
<tr>
<td>$B^+$ (5279)</td>
<td>$D^\pm +..., D^0 +...$</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>⚫</td>
</tr>
<tr>
<td>$B^0$ (5279)</td>
<td>$D^\pm +..., D^0 +...$</td>
<td>0</td>
<td>0</td>
<td>+1</td>
<td>⚫</td>
</tr>
<tr>
<td>$\Upsilon$ (9460)</td>
<td>Various</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>Self</td>
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</tbody>
</table>
(1) Which of the follow statements are true:

- I: The Standard Model of particle physics describes three fundamental forces of nature: strong, weak, and electromagnetic.
- II: Quantum field theory (special relativity + quantum mechanics) predicts that every particle has an otherwise identical anti-particle but with opposite electric charge.
- III: All of the fundamental particles in the Standard Model are either spin-$\frac{1}{2}$ or spin-1.

(a) I, II, and III are all true.
(b) I and II are true, III is false. ⇐
(c) I is true, II and III are false.
(d) II is true, I and III are false.
(e) II and III are true, I is false.

(2) Which of the following does not involve an experimental confirmation of general relativity:

(a) Perihelion shift of Mercury
(b) Taylor-Hulse binary pulsar system
(c) Gravitational lensing
(d) Black hole evaporation ⇐
(e) Gravitational wave detection

(3) Which of these microscopy techniques has the highest possible resolution:

(a) scanning tunneling ⇐
(b) ultraviolet
(c) optical
(d) magnetic force
(e) piezoelectric
(4) Approximately what is the total volume of a gold nucleus, $^{197}_{79}$Au?

(a) $1.5 \times 10^2 \text{fm}^3$
(b) $5.7 \times 10^2 \text{fm}^3$
(c) $6.1 \times 10^2 \text{fm}^3$
(d) $8.3 \times 10^2 \text{fm}^3$
(e) $1.4 \times 10^3 \text{fm}^3$

(5) At the LHC, $^{208}_{82}$Pb nuclei are accelerated and collide at a center-of-mass energy of 5500 GeV per nucleon. How much do the nuclei “flatten” (i.e., what is the ratio of the diameter of the Pb nucleus along the direction of travel to the diameter orthogonal to the direction of travel)?

(a) $1.7 \times 10^{-4}$
(b) $2.4 \times 10^{-4}$
(c) $3.4 \times 10^{-4}$
(d) $6.8 \times 10^{-4}$
(e) $8.7 \times 10^{-4}$

(6) Which of the following statements are true about baryogenesis:
- This process happens while the Universe is sufficiently hot that nucleons cannot bind to form light nuclei.
- The CP violation present in the Standard Model is insufficient to account for the matter-anti-matter asymmetry.
- Heavy ion collisions at the LHC do not reach temperatures as high as the Universe experiences during this time.

(a) I, II, and III are all true.
(b) I and II are true, III is false.
(c) I and III is true, II is false.
(d) II is true, I and III are false.
(e) II and III are true, I is false.
(7) Identify which of the following reactions is \textit{strangeness violating}:

(a) \( K^+ + K^0 \rightarrow \Lambda + \Lambda + \pi^+ \)
(b) \( K^- + p \rightarrow \Omega^- + K^0 + K^+ \)
(c) \( e^+ + e^- \rightarrow J/\psi + \pi^0 + \eta^0 \)
(d) \( K^- + p \rightarrow \Xi^+ + \Omega^- + n + \pi^0 \)
(e) \( D_s^+ \rightarrow K^+ + K^- + \pi^+ \)

(8) Identify which of the following reactions is \textit{valid}:

(a) \( p \rightarrow n + e^+ + \nu_e \)
(b) \( \pi^0 \rightarrow \mu^+ + \mu^- \)
(c) \( \Xi^+ \rightarrow K^+ + K^0 + \pi^+ + \pi^- \)
(d) \( B^+ \rightarrow J/\psi + p + \bar{p} + \pi^+ \)
(e) \( \Sigma^0 \rightarrow \Lambda + \pi^0 \)

(9) Which of the following statements are \textit{true}?

- The heaviest nucleus that the solar fusion process ends with is Boron-8.
- Neutrinos produced in the proton-proton cycle arrive at the Earth’s surface before the photons do.
- The mean core temperature of the Sun corresponds to an average kinetic energy that is lower than what is needed for protons to overcome the Coulomb barrier.

(a) I, II, and III are all true.
(b) I and II are true, III is false.
(c) I and III is true, II is false.
(d) II is true, I and III are false.
(e) II and III are true, I is false.
(10) Which of the following statements are true?

(a) When uranium undergoes fission, energy is released because uranium has more binding energy per nucleon than do the fission products.
(b) Energy is released during deuteron fusion into helium because the deuterons have more binding energy per nucleon than do helium nuclei.
(c) The mass of a stable nucleus is more than the sum of the masses of the nucleons inside it.
(d) The mass of a stable nucleus equals the sum of the masses of the nucleons inside it.
(e) None of these are true. ⇐

(11) Which of the following statements about the strong nuclear force is true?

(a) It is a short range force, reaching only $\sim 10^{-9}$ m.
(b) It is spin dependent. ⇐
(c) It is electric charge dependent.
(d) It causes radioactivity to occur.
(e) It surrounds us, it penetrates us, it binds the galaxy together.

(12) Given three flavors of quarks (e.g. $u$, $d$, $s$), how many different ways can they be combined to give a unique set of baryons?

(a) 8
(b) 10 ⇐
(c) 12
(d) 14
(e) 27
(13) Which of the following was not a prediction of Pauli regarding the neutrino?

(a) It has very little mass.
(b) It is electrically neutral.
(c) It has spin $\frac{1}{2}$.
(d) It comes in three flavors. ⇐
(e) It obeys the Pauli exclusion principle.

(14) The Lyman alpha line of hydrogen has a wavelength of 122 nm in the laboratory. If a galaxy is $5 \times 10^9$ light years from us, what wavelength would we expect to measure for the Lyman alpha line emitted from that galaxy?

(a) 56 nm
(b) 83 nm
(c) 178 nm ⇐
(d) 218 nm
(e) 258 nm

(15) Which of the following statements is false?

(a) The deuteron (proton-neutron bound state) can only have total spin equal to 1.
(b) Alpha particle emission can be understood as a quantum (tunneling) phenomenon.
(c) The proton-proton bound state plays an important role in solar fusion. ⇐
(d) Gamma decay involves the de-excitation of nuclei.
(e) Large $A$ nuclei are destabilized by the Coulomb repulsion between the protons in the nucleus.
(16) Which process below does not involve a weak interaction?

(a) \( n \rightarrow p + e^- + \bar{\nu}_e \)

(b) \( \Sigma \rightarrow \Lambda + \gamma \)

(c) \( p + p \rightarrow d + e^+ + \nu_e \)

(d) \( \Sigma^- \rightarrow n + \pi^- \)

(e) \( K^+ \rightarrow \pi^+ + \pi^0 \)

(17) Which of the following statements are true about the Universe?

- It has been expanding at a constant rate for roughly 13.6 billion years.
- Matter dominates over anti-matter by about one part in \( 10^9 \).
- The cosmic microwave background originates with photons that no longer energetic enough to ionize electrons from nuclei.

(a) I, II, and III are all true.

(b) I and II are true, III is false.

(c) I and III is true, II is false.

(d) II is true, I and III are false.

(e) II and III are true, I is false.

(18) Which of the following statements is false about gravitational wave detection?

(a) The LIGO detectors are essentially very large interferometers.

(b) Gravitational waves squeeze and stretch spacetime.

(c) The first observation of gravitational waves likely originated from rapidly orbiting black holes or neutron stars.

(d) Three separate locations measured the gravitational waves to triangulate the source’s position in the sky.

(e) One reason this observation is important is because astronomers now have a new technique for observing the cosmos.
(19) Which absolute conservation law does the process $\gamma \rightarrow e^+e^-$ violate?

(a) Energy/momentum ⇐
(b) Angular momentum
(c) Baryon number
(d) Lepton ($L_e$) number
(e) Electric charge

(20) Which Rutgers Professor did not give a lecture in this class?

(a) Chou
(b) Gawiser ⇐
(c) Lath
(d) Salur
(e) Wu
(21) Bonus Question 1: In 1943 in the French Alps, two French physicists took $\sim 10,000$ triggered photographs of cosmic rays showers in a cloud chamber immersed in a uniform $0.245$ T magnetic field. They found one photo (shown below) in which a particle with initial momentum $p$ but unknown mass $M$ scatters an electron (initially at rest) at an incidence angle $\theta$ with respect to the initial direction of motion of the incident particle.

(a) What is $M$ in terms of $\theta$, $p$, $m_e$, and the electron’s outgoing energy $E$?

(b) Based on the magnetic field and the radii of curvature, the momenta of the incident particle and the electron were determined to be approximately $51$ MeV/$c$ and $0.11$ MeV/$c$, respectively. While the incident particle was negligibly scattered by the collision, the electron was scattered at an angle $\theta = 20^\circ$. Assuming that the particle has the elementary charge $e$, what is your best guess for the identity of this particle?

Answer: Using four vectors is not strictly necessary here since we are not evaluating the system in multiple frames. In the lab frame, momentum conservation (in the longitudinal direction) requires that

$$p = k \cos \phi + p_e \cos \theta,$$

where $k$ is the momentum of the incident particle after the collision, $\phi$ is the scattered angle with respect to the initial trajectory, and $p_e$ is the magnitude of the momentum of the electron. Momentum conservation (in the transverse direction) also requires that

$$k \sin \phi = p_e \sin \theta.$$
Energy conservation requires that
\[ \sqrt{p^2c^2 + M^2c^4 + m_ec^2} = \sqrt{k^2c^2 + M^2c^4} + E. \]
We can solve for \( k^2 \) and remove the reference to \( \phi \) by re-arranging and squaring the first momentum conservation equation,
\[ (p - p_e \cos \theta)^2 = k^2 \cos^2 \phi, \]
and adding to it the square of the second momentum conservation equation
\[ (p - p_e \cos \theta)^2 + p_e^2 \sin^2 \theta = k^2 \cos^2 \phi + k^2 \sin^2 \phi, \]
and so
\[ k^2 = p^2 + p_e^2 - 2pp_e \cos \theta. \]
We can plug this result into the energy conservation equation to get
\[ \sqrt{p^2c^2 + M^2c^4 + m_ec^2} = \sqrt{p^2c^2 + p_e^2c^2 - 2pp_e c^2 \cos \theta + M^2c^4} + E. \]
Moving \( E \) from the RHS to the LHS and squaring gives
\[ p^2c^2 + M^2c^4 + (E - m_ec^2)^2 - 2(E - m_ec^2) \sqrt{p^2c^2 + M^2c^4} = p^2c^2 + p_e^2c^2 - 2pp_e c^2 \cos \theta + M^2c^4. \]
The \( p^2c^2 \) and \( M^2c^4 \) terms on both sides cancel, and we can rearrange terms to give
\[ \sqrt{p^2c^2 + M^2c^4} = \frac{p_e^2c^2 - 2pp_e c^2 \cos \theta - (E - m_ec^2)^2}{-2(E - m_ec^2)}. \]
Noting that \( E^2 = p_e^2c^2 + m_ec^4 \), we get
\[ \sqrt{p^2c^2 + M^2c^4} = \frac{E^2 - m_ec^4 - 2pc \sqrt{E^2 - m_ec^4 \cos \theta - E^2 - m_ec^4} + 2Em_ec^2}{-2(E - m_ec^2)} \]
\[ = \frac{pc \sqrt{E^2 - m_ec^4 \cos \theta - Em_ec^2 + m_ec^4}}{E - m_ec^2} \]
\[ = pc \cos \theta \sqrt{\frac{E + m_ec^2}{E - m_ec^2}} - m_ec^2. \]
Solving for \( M \) gives
\[ M = \frac{p}{c} \left[ \cos \theta \sqrt{\frac{E + m_ec^2}{E - m_ec^2}} - \frac{m_ec^2}{pc} \right]^\frac{1}{2}, \]
which we can alternatively write as
\[ M = \frac{p}{c^2} \left[ \left( \cos \theta \sqrt{\frac{E + m_ec^2}{E - m_ec^2}} - \frac{m_ec^2}{pc} \right)^2 - p^2c^2, \right]^\frac{1}{2}. \]
Since we are interested in the case where $pc \gg mc^2$, this could be numerically approximated as

$$M \approx \frac{p}{c} \left[ \frac{E + m^2c^2}{E - m^2c^2} \cos^2 \theta - 1 \right]^{\frac{1}{2}}.$$  

Plugging in $p = 51$ MeV/c, $E = \sqrt{0.11^2 + 0.511^2} \approx 0.523$ MeV, and $\cos \theta \approx 0.940$ to the above equation gives $M \approx 460$ MeV/c^2. This is approximately the mass of the charged kaon, $K^\pm$.  

(22) Bonus Question 2: Recall that the time independent Schrödinger equation for the 1D simple harmonic oscillator can be re-expressed as

\[(a_+a_- + \frac{1}{2} \hbar \omega)\psi_n = E_n \psi_n,\]

where \(n\) is an integer, \(E_n = (n + \frac{1}{2}) \hbar \omega\), and the raising and lowering operators are defined as

\[a_\pm \equiv \frac{1}{\sqrt{2m}} \left( \frac{\hbar}{i} \frac{d}{dx} \pm im\omega x \right).\]

The solutions are

\[\psi_n(x) = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} (-i)^n \sqrt{n!(h\omega)^n} (a_+)^n \exp \left( -\frac{m\omega}{2\hbar} x^2 \right).\]

Let a particle be in a linear superposition of the ground state and an arbitrary parameterization of the first excited state at an instantaneous time \(t = 0\):

\[\Psi(x, 0) = A[\psi_0(x) + B\psi_1(x)],\]

where \(B\) is an arbitrary real constant such that \(B \geq 0\).

(a) What is \(A\) (in terms of \(B\))? Calculate \(\psi_0\) and \(\psi_1\). Write down the analytic expression for \(\Psi(x, t)\) (no derivatives, raising/lowering operators, etc.).

(b) Calculate \(|\Psi(x, t)|^2\) without recourse to imaginary numbers.

(c) Calculate \(\langle x \rangle\) and \(\langle p \rangle\). Recall that

\[\int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \sqrt{\frac{\pi}{\alpha}},\]

\[\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{1}{2} \sqrt{\frac{\pi}{\alpha^3}}.\]

(d) For what value of \(B\) is the time dependence of \(\langle x \rangle\) and \(\langle p \rangle\) minimized? Maximzed?

Answer: Since \(\psi_0\) and \(\psi_1\) are already orthonormal,

\[\int |\Psi(x, 0)|^2 dx = A^2 \int \left( |\psi_0|^2 + 2B\psi_0^\ast \psi_1 + B^2 \psi_0^2 + 2B\psi_0 \psi_1^\ast \right) dx = A^2(1 + B^2) = 1,\]

so \(A = \sqrt{\frac{1}{1 + B^2}}\). Applying the time dependent Schrödinger equation to the wavefunction gives

\[\Psi(x, t) = \sqrt{\frac{1}{1 + B^2}} (\psi_0(x)e^{-iE_0 t/\hbar} + \psi_1(x)e^{-iE_1 t/\hbar}).\]

Using the equations given above, we can calculate the stationary states \(\psi_0\) and \(\psi_1\):

\[\psi_0(x) = \left( \frac{m\omega}{\pi \hbar} \right)^{1/4} \exp \left( -\frac{m\omega}{2\hbar} x^2 \right).\]
and

\[ \psi_1(x) = \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} \left( -i \right) \frac{1}{\sqrt{\hbar \omega}} \frac{h}{i} \frac{d}{dx} + im \omega x \exp \left( -\frac{m \omega}{2 \hbar} x^2 \right) \]

\[ = \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} \left( -i \right) \frac{1}{\sqrt{\hbar \omega}} \frac{h}{i} \frac{m \omega}{\hbar} x + im \omega x \exp \left( -\frac{m \omega}{2 \hbar} x^2 \right) \]

\[ = \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} \sqrt{\frac{2m \omega}{\hbar}} \frac{1}{\hbar} x \exp \left( -\frac{m \omega}{2 \hbar} x^2 \right). \]

Plugging these in to the expression for \( \Psi(x,t) \) above as well as \( E_0 = \frac{1}{2} \hbar \omega \) and \( E_1 = \frac{3}{2} \hbar \omega \) we get

\[
\Psi(x,t) = \sqrt{\frac{1}{1 + B^2}} \left( \frac{m \omega}{\pi \hbar} \right)^{1/4} \exp \left( -\frac{m \omega}{2 \hbar} x^2 \right) \left[ \exp \left( -\frac{1}{2} i \omega t \right) + B \sqrt{\frac{2m \omega}{\hbar}} x \exp \left( -\frac{3}{2} i \omega t \right) \right].
\]

Multiplying this expression by its complex conjugate gives

\[
|\Psi(x,t)|^2 = \frac{1}{1 + B^2} \left( \frac{m \omega}{\pi \hbar} \right)^{1/2} \exp \left( -\frac{m \omega}{\hbar} x^2 \right) \left[ 1 + B^2 \frac{2m \omega}{\hbar} x^2 + 2B \sqrt{\frac{2m \omega}{\hbar}} x \cos(\omega t) \right].
\]

The expectation value of \( x \) is

\[
\langle x \rangle = \int_{-\infty}^{\infty} \Psi^* x \Psi dx = \int_{-\infty}^{\infty} x |\Psi|^2 dx
\]

\[ = \frac{1}{1 + B^2} \left( \frac{m \omega}{\pi \hbar} \right)^{1/2} \int_{-\infty}^{\infty} \exp \left( -\frac{m \omega}{\hbar} x^2 \right) \left[ x + \frac{2m \omega}{\hbar} B^2 x^2 + 2B \sqrt{\frac{2m \omega}{\hbar}} x^2 \cos(\omega t) \right] dx
\]

\[ = \frac{1}{1 + B^2} \left( \frac{m \omega}{\pi \hbar} \right)^{1/2} \int_{-\infty}^{\infty} \exp \left( -\frac{m \omega}{\hbar} x^2 \right) 2B \sqrt{\frac{2m \omega}{\hbar}} x^2 \cos(\omega t) dx
\]

\[ = \frac{2B}{1 + B^2} \left( \frac{m \omega}{\pi \hbar} \right)^{1/2} \sqrt{\frac{2m \omega}{\hbar}} \cos(\omega t) \frac{1}{2} \sqrt{\frac{\pi \hbar^3}{m^3 \omega^3}}
\]

\[ = \frac{B}{1 + B^2} \sqrt{\frac{\hbar}{2m \omega}} \cos(\omega t). \]

We can use Ehrenfest’s theorem to get the expectation value of \( p \):

\[
\langle p \rangle = m \frac{d\langle x \rangle}{dt} = -\frac{B}{1 + B^2} \sqrt{\frac{\hbar m \omega}{2}} \sin(\omega t).
\]
The amplitude of modulation is minimized (equal to 0) when $B \to 0$ and $B \to \infty$, i.e. when the wavefunction is either pure $\psi_0$ or $\psi_1$. It is maximized when

$$\frac{d}{dB} \left( \frac{B}{1 + B^2} \right) = \frac{1 + B^2 - 2B^2}{(1 + B^2)^2} = 0 \rightarrow B = 1,$$

where the negative solution is dropped since $B$ is defined to be $\geq 0$. 