The (anomalous) Zeeman effect
Since the electron is orbiting the proton, it’s movement constitutes a current. Let’s pretend it’s moving in a circle.

The period of one orbit is \( t = \frac{2\pi r}{v} \)

So the current is \( I = \frac{q}{t} = \frac{-e}{2\pi r/v} = -\frac{ev}{2\pi r} \)

And any current has a magnetic moment:

\[
\mu = I \cdot A = -\frac{ev}{2\pi r} \cdot \pi r^2 = -\frac{evr}{2}
\]

Therefore

\[
\frac{\mu}{L} = -\frac{evr/2}{m_e vr} = -\frac{e}{2m_e} \quad \rightarrow \quad \vec{\mu} = -\frac{e}{2m_e} \vec{L}
\]
For convenience, let’s introduce a constant $g$ called the “gyromagentic-factor” or “g-factor”, where $g=1$.

Then we have:

$$\vec{\mu} = -g \frac{e}{2m_e} \vec{L}$$

When a magnetic moment is placed in an external magnetic field in the $z$ direction, the potential energy of the interaction is:

$$\Delta E = -\vec{\mu} \cdot \vec{B} = - \left( -g \frac{e}{2m_e} \right) \vec{L} \cdot \vec{B} = g \frac{eB}{2m_e} L_z$$

Substituting for $L_z$, we get

$$\Delta E = g \frac{eB}{2m_e} (m\hbar) = mg\mu_B B \text{ where } \mu_B \equiv \frac{e\hbar}{2m_e} \approx 5.8 \times 10^{-5} \text{ eV/T}$$
By applying an external magnetic field to the Hydrogen atom, the levels should be split according to their **magnetic quantum number** \( m \). This is the “Normal” Zeeman effect. (This is not what actually happens)

\[
\Delta E = mg\mu_B B \text{  where  } m = 0, \pm 1, \pm 2, \ldots
\]

<table>
<thead>
<tr>
<th>Energy Level (eV)</th>
<th>Quantum Level</th>
<th>Magnetic Quantum Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>-13.6/9</td>
<td>3s</td>
<td>0</td>
</tr>
<tr>
<td>-13.6/4</td>
<td>2s</td>
<td>0</td>
</tr>
<tr>
<td>-13.6</td>
<td>1s</td>
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<td>+1</td>
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<tr>
<td></td>
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<td>-1</td>
</tr>
</tbody>
</table>

-13.6/9 eV: 3s → 0
-13.6/4 eV: 2s → 0
-13.6 eV: 1s → 0
In an external B-field, every photon line splits into exactly 3 equally-spaced lines (photon has $S=1$). This is the “normal” Zeeman effect.

- The size of the spacing is $\Delta E = g\mu_B \cdot B$.
- Experimentally, the Zeeman effect does occur, but for many elements (such as in Hydrogen) it is not “normal” in that the spacing is not $g\mu_B \cdot B$. This is known as the “anomalous” Zeeman effect.
- Photon lines can often be split into more than three in an external B-field.
- Even when $B=0$, there can still be observed a splitting.
The spin $S$ of the electron has many similarities to the orbital angular momentum $L$.

- It has an associated quantum number $s$, which takes on a value of $1/2$. It has a z component $m_s$ which is $\pm 1/2$.
- The total spin is determined by the equation:

$$|\vec{S}| = \hbar \sqrt{s(s+1)} = \sqrt{\frac{3}{4}} \hbar$$

- The z-component of the spin is

$$S_z = m_s \hbar = \pm \frac{1}{2} \hbar$$

- The magnetic moment of the electron is

$$\vec{\mu}_s = -g \frac{e}{2m} \vec{S} \text{ with } g = 2 \text{ (!) }$$
Electron magnetic moment is one of the most accurately measure properties of an elementary particle and one of the properties of particle that can be most accurately predicted by the standard model

It is an essential part for what is arguably the most stringent tests of the standard model

$g \sim 2$ comes from considerations of the Dirac equation (the “relativistic Schrödinger equation”, if you like).

measurement:
$g/2 = 1.001\ 159\ 652\ 180\ 73(28)$

theory:
$g/2 = 1.001\ 159\ 652\ 181\ 64(76)$
Consider an electron in a magnetic field \( B \) along the \( z \) axis. In classical electromagnetism, a **magnetic moment** in a \( B \)-field experiences a torque

\[
\vec{\tau} = \vec{\mu} \times \vec{B}
\]

that **tends to align the dipole with the magnetic field**.

The electron can never be spinning with its magnetic moment exactly along the \( z \) axis.
Given the strong similarities between them, we can also try **adding together** orbital angular momentum and spin.

The total angular momentum $J$ also behaves like quantized angular momentum:

$$J = |\vec{J}| = \sqrt{j(j + 1)}\hbar$$

$$J_z = m_j\hbar$$

But since $m_\ell$ is integral and $m_s$ is half-integral, $m_j$ must also be half-integral.

And just as $m_\ell$ ranged from $-\ell$ to $+\ell$, $m_j$ ranges from $-j$ to $j$. 
When forming the total angular momentum, the addition of \( L \) and \( S \) has to be done \textbf{vectorially}. Schematically, this looks like:

```
\[
\begin{array}{c}
\text{S} \\
\text{L} \\
\text{J}
\end{array}
\text{J}
\begin{array}{c}
\text{S} \\
\text{L} \\
\text{J}
\end{array}
\text{L}
```

Problem for quantum mechanical angular momentum: \textbf{we can’t know} \( L_x \) \textbf{and} \( L_y \) \textbf{simultaneously with} \( L_z \) (and similarly for \( J \) and \( S \)).

The total angular momentum can \textbf{only have the values} \( j = \ell \pm s \) (except if \( \ell = 0 \), in which case \( j = 1/2 \)).
Spin-Orbit Coupling

In the rest-frame of the electron (nevermind that it is not inertial for now), the proton is revolving about it. This means that the proton has a magnetic moment about the electron.

This means that the magnetic moment of the electron (proportional to $S$) interacts with the magnetic moment of the proton (proportional to $L$). This is the spin-orbit coupling.

This results in a fine structure splitting ($\sim 10^{-5}$ eV) of the Hydrogen energy levels:

$$E_{\text{fine}} = E_{\text{Bohr}} \left[ 1 + \frac{\alpha^2}{n} \left( \frac{1}{j + 1/2} - \frac{3}{4n} \right) \right]$$

Degeneracy between levels with the same $n$ and $j$. 
Useful to use spectroscopic notation: \( n^{2s+1} L_j \)

remember that \( j=|\ell - s|, \ldots, (\ell + s) \).

note 1: not to scale

note 2: levels of the same \( n \) and \( j \) are degenerate
In addition to spin-orbit interactions, there is also spin-spin interactions (for atoms with multiple electrons), as well as orbit-orbit interactions.

The spin of the nucleus can also induce hyperfine splitting, but this is a $6 \times 10^{-6}$ eV effect in hydrogen (only seen in astronomic measurements): 21cm line

Carl Sagan thought this a sufficiently universal effect that the units of time and length on the Pioneer spacecraft are based on it.
At this point we have calculated two different magnetic moments of the electron.

One is due to its rotation about the nucleus:

$$\vec{\mu}_L = -g \frac{e}{2m} \vec{L} \text{ with } g = 1$$

Because of this magnetic moment, we deduced that by putting an atom in a magnetic field we would get splittings according to the quantum number $m_\ell$:

$$\Delta E = g \frac{eB}{2m_e} (m_\ell \hbar) = m_\ell g \mu_B B \text{ where } \mu_B \equiv \frac{e\hbar}{2m_e} \approx 5.8 \times 10^{-5} \frac{\text{eV}}{\text{T}}$$

hence the name: $m_\ell$ is the “magnetic” quantum number.
Another magnetic moment is due to the \textit{intrinsic spin} of the electron:

\[ \vec{\mu}_s = -g \frac{e}{2m} \vec{S} \text{ with } g = 2 \ (! \ ) \]

Here we find that the interaction between the \textit{spin} magnetic moment of the electron and the “magnetic field” of the nucleus creates \textit{fine-structure splitting}.

The interaction between the electron and the nucleus is called a \textit{spin-orbit coupling}.
Now let’s put these two together: spin-orbit effects in a magnetic field.

In order to do this, we need to calculate $\mu_J$ (not just $\mu_L$), that is, the magnetic moment from the total angular momentum.

$$\vec{\mu}_L = -g \frac{e}{2m} \vec{L} \text{ with } g = 1$$

$$\vec{\mu}_S = -g \frac{e}{2m} \vec{S} \text{ with } g = 2$$

Because $g=2$, $\mu_S+\mu_L \neq \mu_J$:

See how $S$ is anti-parallel to $\mu_S$, and $L$ in anti-parallel to $\mu_L$, but $J$ is not anti-parallel to $\mu_J$?
In order to deal with this, we will define $\mu_J$ as the **total magnetic moment** (adding orbital and spin components together) along the J direction:

$$\vec{\mu}_J = -g \frac{e}{2m_e} \vec{J}$$

where “g” is the Lande g factor:

$$g = 1 + \frac{j(j + 1) + s(s + 1) - \ell(\ell + 1)}{2j(j + 1)}$$

This g-factor is an effective constant that compensates for the fact that the magnetic moment is not anti-parallel to the J direction.
Just as with the “normal” Zeeman effect, if you place the atom in a magnetic field, you get an energy shift

\[ \Delta E = g \frac{eB}{2m_e} (m_j \hbar) = m_j g \mu_B B \]

but this time, it’s proportional to \( j \) (not \( \ell \)), and the \( g \)-factor depends on \( j \), \( s \), and \( \ell \). This is the anomalous Zeeman effect.

\[ \Delta E = \frac{1}{3} \mu_B B \]

\[ \Delta E = \mu_B B \]