Spin-Orbit Coupling and the Pauli Exclusion Principle
Electrons have a property known as **intrinsic spin**: Analogous to a particle spinning about it’s own axis:

Suppose the electron were an **extended object** with electric charge. If it possessed angular momentum, that would imply an electric current, which would induce a magnetic moment (remember the magnetic moment of the Hydrogen atom!)

In reality, the electron is not extend. It is a **point-like particle**. But it still possesses the fundamentally quantum mechanical property of “spin.”
The spin $S$ of the electron has many similarities to the orbital angular momentum $L$.

- It has an associated quantum number $s$, which takes on a value of 1/2. It has a z component $m_s$ which is $\pm 1/2$.
- The total spin is determined by the equation:
  \[ |\vec{S}| = \hbar \sqrt{s(s + 1)} = \sqrt{\frac{3}{4}} \hbar \]
- The z-component of the spin is
  \[ S_z = m_s \hbar = \pm \frac{1}{2} \hbar \]
- The magnetic moment of the electron is
  \[ \vec{\mu}_s = -g \frac{e}{2m} \vec{S} \text{ with } g = 2 \text{ (!)} \]
Electron magnetic moment is one of the most accurately measure properties of an elementary particle and one of the properties of particle that can be most accurately predicted by the standard model.

It is an essential part for what is arguably the most stringent tests of the standard model.

Measurement:
\[
g/2 = 1.001\ 159\ 652\ 180\ 73(28)\]

Theory:
\[
g/2 = 1.001\ 159\ 652\ 181\ 64(76)\]
Consider an electron in a magnetic field $B$ along the $z$ axis. In classical electromagnetism, a **magnetic moment** in a $B$-field experiences a torque

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

that **tends to align the dipole with the magnetic field.**

The electron can never be spinning with its magnetic moment exactly along the $z$ axis.
Given the strong similarities between them, we can also try \textbf{adding together} orbital angular momentum and spin.

The total angular momentum $J$ also behaves like quantized angular momentum:

$$J = |\vec{J}| = \sqrt{j(j + 1)}\hbar$$

$$J_z = m_j \hbar$$

But since $m_\ell$ is integral and $m_s$ is half-integral, $m_j$ must also be half-integral.

And just as $m_\ell$ ranged from $-\ell$ to $+\ell$, $m_j$ ranges from $-j$ to $j$. 
When forming the total angular momentum, the addition of \( L \) and \( S \) has to be done \textit{vectorially}. Schematically, this looks like:

Problem for quantum mechanical angular momentum: we can’t know \( L_x \) and \( L_y \) simultaneously with \( L_z \) (and similarly for \( J \) and \( S \)).

The total angular momentum can \textbf{only have the values} \( j = \ell \pm s \) (except if \( \ell = 0 \), in which case \( j = 1/2 \)).
Spin-Orbit Coupling

In the rest-frame of the electron (nevermind that it is not inertial for now), the proton is revolving about it. This means that the proton has a magnetic moment about the electron.

This means that the magnetic moment of the electron (proportional to $S$) interacts with the magnetic moment of the proton (proportional to $L$). This is the spin-orbit coupling.

This results in a **fine structure splitting** (~$10^{-5}$ eV) of the Hydrogen energy levels:

$$E_{\text{fine}} = E_{\text{Bohr}} \left[ 1 + \frac{\alpha^2}{n} \left( \frac{1}{j + 1/2} - \frac{3}{4n} \right) \right]$$

Degeneracy between levels with the same $n$ and $j$. 

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$\alpha$ is the fine structure constant.
Useful to use spectroscopic notation: \( n^{2s+1} L_j \)

remember that \( j = |\ell - s|, \ldots, (\ell + s) \).

\[ \begin{align*}
4^2 S_{1/2} & \quad \quad 4^2 P_{3/2} \\
3^2 S_{1/2} & \quad \quad 3^2 P_{1/2} \\
2^2 S_{1/2} & \quad \quad 2^2 P_{1/2} \\
1^2 S_{1/2} & \\
\end{align*} \]

note 1: not to scale

note 2: levels of the same \( n \) and \( j \) are degenerate
PRS Question: Which of the following statements are true about the H atom fine-structure in no external field.
A) $3P_{3/2}$ is degenerate with $3D_{3/2}$ but not with $3P_{1/2}$
B) $3P_{3/2}$ is degenerate with $3P_{1/2}$ but not with $3D_{3/2}$
C) $3P_{3/2}$ is degenerate with $3D_{3/2}$ as well as $3P_{1/2}$
For fine-structure without an external field, n and j are degenerate.
In addition to spin-orbit interactions, there is also spin-spin interactions (for atoms with multiple electrons), as well as orbit-orbit interactions.

The spin of the nucleus can also induce hyperfine splitting, but this is a $6 \times 10^{-6}$ eV effect in hydrogen (only seen in astronomic measurements): 21cm line

Carl Sagan thought this a sufficiently universal effect that the units of time and length on the Pioneer spacecraft are based on it.
Pauli Exclusion Principle (1925)

No two electrons in an atom may have the same set of quantum numbers, \( n, \ell, m_\ell, m_s \).

Ground state of helium:
Electron #1: \( n=1, \ell=0, m_\ell=0, m_s=+1/2 \) (or -1/2)
Electron #1: \( n=1, \ell=0, m_\ell=0, m_s=-1/2 \) (or +1/2)

Ground state of lithium:
Electrons 1 and 2 are like helium.
Electron #3 \( n=2, \ell=0, m_\ell=0, m_s=+1/2 \) or -1/2
For each \( n \), \( \ell = 0, 1, \ldots, (n-1) \). So the maximum shell population is \( 2n^2 \).
PRS Question: What is the maximum number of electrons that can populate a 3d subshell?
A) 5
B) 6
C) 10
D) 14
E) None of the above
“d” means $\ell = 2$.

So $m_{\ell}$ can be -2, -1, 0, 1, 2 (5 possibilities)
And there are two electron spin possibilities. $5 \times 2 = 10$. 

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