Schrodinger’s Equation

Alice’s Adventures in Wonderland, Chapter VI: The Cheshire Cat gets Weirder.
Fitting the distribution to a Gaussian gives a mean of 10 and width of 3.3.
Exam question 8

A beam of light is emitted at an angle of 30° with respect to the x-axis in the rest frame. Another inertial frame moving along the x-axis in the positive direction observes that the angle is 90°. What is the velocity of the frame?

a) About 0.87c (26%)
b) About 0.90c (0%)
c) About 0.94c (8%)
d) About 1.0c (17%)
e) Impossible (50%)
Exam question 8

A beam of light is emitted at an angle of 30° with respect to the x-axis in the rest frame. Another inertial frame moving along the x-axis in the positive direction observes that the angle is 90°. What is the velocity of the frame?

\[
\begin{align*}
\mathbf{u}'_x &= \frac{u_x - v}{1 - vu_x/c^2} \\
\mathbf{u}'_y &= \frac{u_y}{\gamma[1 - vu_x/c^2]}
\end{align*}
\]

\[
\begin{align*}
\mathbf{u}'_z &= \frac{u_z}{\gamma[1 - vu_x/c^2]}
\end{align*}
\]

\[u_x = c \cos(\theta)\] and \[u'_x = 0\], therefore \[v = u_x = c \cos(30°) = 0.87c\]
Exam question 16

Electrons at rest are hit by incident photons. The maximum kinetic energy given to an electron is 30 keV. What is the wavelength of the light?

a) 0.0243 nm (8%)
b) 0.486 nm (30%)
c) 0.030 nm (6%)
d) 0.012 nm (24%)
e) None of the above (30%)
Exam question 16

Electrons at rest are hit by incident photons. The maximum kinetic energy given to an electron is 30 keV. What is the wavelength of the light?

from energy conservation: \( \frac{hc}{\lambda} + mc^2 = \frac{hc}{\lambda'} + K + mc^2 \)

from Compton scattering: \( \lambda' - \lambda = \frac{h}{mc} (1 - \cos \theta) \)

we want the max kinetic energy, so \( \theta = \pi \), therefore: \( \lambda' = \lambda + \frac{2h}{mc} \)

plug into energy cons. to get: \( \frac{K}{hc} = \left( \frac{1}{\lambda} - \frac{1}{\lambda + \frac{2h}{mc}} \right) \)

Plugging in #’s gives: \( 24.2 = \left( \frac{1}{\lambda} - \frac{1}{\lambda + 0.0048} \right) \)

This is satisfied by \( \lambda = 0.012 \text{ nm} \)
Quantum Mechanics is a fundamental shift from a deterministic theory (Newtonian mechanics, special relativity, etc.) to a **probabilistic theory**.

The problem with Bohr’s theory of the Hydrogen atom is that he retains too many classical elements. He needed to disband with determinism altogether.

The major historical breakthrough was Schrödinger’s wave equation. A particle should be expressed as a **complex probability density** $\psi(x,t)$, known as the **wavefunction**.

The probability to find a particle at a given time $t$ between $x_1$ and $x_2$ is given by:

$$P(t) = \int_{x_1}^{x_2} \psi^*(x,t) \psi(x,t) \, dx$$
The wavefunction must be a solution to the Schrödinger equation:

\[ i\hbar \frac{\partial \Psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + V(x, t)\Psi(x, t) \]

This is the one-dimensional, time-dependent version. Below is the three-dimensional version:

\[ i\hbar \frac{\partial \Psi(\vec{x}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{x}, t) + V(\vec{x}, t)\Psi(\vec{x}, t) \]

where

\[ \nabla \equiv \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]
If the potential \( V \) is independent of time, we can use the **time-independent Schrodinger equation**:

\[
-\frac{\hbar^2}{2m} \nabla^2 \psi(x) + V(x) \psi(x) = E \psi(x)
\]

To go to one dimension just replace the Laplacian with a second-order partial derivative:

\[
\nabla^2 \rightarrow \frac{\partial^2}{\partial x^2}
\]

Here, \( E \) is the energy of the particle.
Since the wavefunction is **fundamentally probabilistic**, here are some of the properties it must have:

- At any given time, the total probability over all space must be unity:
  \[
  \int_{-\infty}^{\infty} \Psi^*(\vec{x}, t) \Psi(\vec{x}, t) d^3\vec{x} = 1
  \]

- In order to avoid infinite probabilities, the wavefunction must be finite everywhere.

- It must be single-valued (unique probability values).

- For finite potentials, \( \psi \) and its derivative must be continuous.

- In order to be normalizable (i.e. have a finite integral), \( \psi \) must approach zero as \( x \) approaches infinity.
Where does Schrodinger’s equation come from? It turns out it’s the (somewhat miraculous) replacements:

$$\vec{p} \rightarrow \frac{\hbar}{i} \nabla = -i\hbar \nabla$$  \hspace{1cm} E \rightarrow i\hbar \frac{\partial}{\partial t}$$

If you put back in the Classical expressions for momentum and energy you go from

$$i\hbar \frac{\partial \Psi(\vec{x}, t)}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \Psi(\vec{x}, t) + V(\vec{x}, t) \Psi(\vec{x}, t)$$

to

$$E \Psi(\vec{x}, t) = \frac{p^2}{2m} \Psi(\vec{x}, t) + V(\vec{x}, t) \Psi(\vec{x}, t)$$
and

\[ E \Psi(\vec{x}, t) = \frac{p^2}{2m} \Psi(\vec{x}, t) + V(\vec{x}, t) \Psi(\vec{x}, t) \]

looks a lot like **Newtonian energy conservation**!

\[ E = \frac{p^2}{2m} + V = \frac{1}{2}mv^2 + V \]
Notice something strange, though. In quantum mechanics, momentum and position depend on their order in an equation:

\[ p_x x \rightarrow -i\hbar \frac{\partial}{\partial x} x = -i\hbar \]

On the other hand...

\[ xp_x \rightarrow -x i\hbar \frac{\partial}{\partial x} \neq -i\hbar \]

Same relationship occurs with energy and time:

\[ Et \rightarrow i\hbar \frac{\partial}{\partial t} t = i\hbar \quad tE \rightarrow ti\hbar \frac{\partial}{\partial t} \neq i\hbar \]
This mathematical oddity reveals a rather startling physical fact.

It’s seems that if you ask for a particle’s position followed by it’s momentum, you get a different answer than if you ask for it in the other order.

Ultimately (and stated without proof), you get the following amazing result:

\[ \Delta x \Delta p_x \geq \frac{\hbar}{2} \]

There is an inverse relationship between how precisely you can measure the position of a particle and it’s momentum (along the same axis). The product is bounded below by Planck’s constant (over 2)!
The physical act of measurement necessarily disturbs the system.

This is true of position and momentum.
It is also true of energy and time.

\[ \Delta t \Delta E \geq \frac{\hbar}{2} \]
Consider the nucleus. A proton is confined to a space of $\Delta x \sim 10^{-15}$ m. What is its minimum kinetic energy?

$$c \Delta p_x \geq \frac{\hbar}{2 \Delta x} = \frac{6.58 \times 10^{-16} \text{ eV} \cdot \text{s}}{2 \cdot 10^{-15} \text{ m}} \cdot 3 \times 10^8 \frac{\text{m}}{\text{s}} \approx 100 \text{ MeV}$$

$$p_{\text{min}} = \frac{100 \text{ MeV}}{c} \rightarrow E_{\text{min}} = \sqrt{100^2 + 938^2} = 943 \text{ MeV}$$

So the minimum kinetic energy is about 5 MeV. This means that the nuclear binding energy must be at least 5 MeV.

Remember that the atomic binding energy for Hydrogen was 13.6 eV!
Suppose if there were electrons present within the nucleus. According to the uncertainty principle, what is the minimum kinetic energy?

As before $p_{\text{min}} = 100 \text{ MeV/c}$, but the energy is different:

$$p_{\text{min}} = 100 \text{ MeV/c} \rightarrow E_{\text{min}} = \sqrt{100^2 + 0.511^2} \approx 100 \text{ MeV}$$

So the binding energy for an electron must be about 100 MeV! Much larger than the binding energy of a proton (or neutron for that matter).

It turns out that the binding energy of the nucleus tops out around 9 MeV/nucleon (nucleon=proton or neutron).