The Bohr Model of Hydrogen
Suppose you wanted to identify and measure the energy of high energy photons. One way to do this is to make a calorimeter.

The CMS experiment’s electromagnetic calorimeter is made of lead crystal (PbWO$_4$). When a high energy photon hits it, the photon interacts with the heavy nucleus, and pair produces. Each electron/positron undergoes bremsstrahlung, which emits more photons, which then pair produce, etc., etc. This results in an electromagnetic shower.
An event display of a single 340 GeV photon in the CMS detector. It is (probably) recoiling against two neutrinos which escape the detector without interacting.
PRS Question: An incident photon strikes an electron that was initially at rest. Which of the following is true about the scattered photon?

A) It’s wavelength equals that of the incident photon
B) It’s wavelength is less than that of the incident photon
C) It’s energy equals that of the incident photon
D) It’s energy is more than that the incident photon
E) None of the above
Compton effect quiz.

From energy conservation:

\[ E_\gamma + m_e c^2 = E_\gamma' + E_e + m_e c^2 \]

\[ E_\gamma > E_\gamma' \]

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Pre-Quantum era questions about the Hydrogen Atom

Why does the Rydberg formula work?
Why is the absorption the same as the emission spectrum?
Why is the ionization energy of Hydrogen 13.6 eV?
Why is the atom stable in the first place?
Recall what would happen if we apply classical electromagnetism to an electron orbiting a proton. Coulomb’s Law binds electron to proton, but the electron is accelerating.

An accelerating electron emits electromagnetic radiation (this is an unavoidable consequence of Maxwell’s equations).

Eventually ($10^{-9}$ seconds) the electron loses enough energy through radiation that is collides with the proton.

\[ F = \frac{q_1 q_2}{4\pi \epsilon_0 r^2} = \frac{mv^2}{r} \]
Bohr introduces his model of the Hydrogen atom in 1913.

**Assumption 1:** the electron can only be in circular orbits whose **orbital angular momentum** is quantized:

\[ L = mvr = n \frac{\hbar}{2\pi} = n\hbar \quad n = 1, 2, 3, \ldots \]

**Assumption 2:** An electron can spontaneously jump to a lower orbit. In doing so, a photon is emitted, whose energy equals the difference in energy between the two orbits.
In Bohr model, Coulomb force still provides centripetal acceleration:

\[
\frac{1}{4\pi\varepsilon_0} \frac{e^2}{r^2} = \frac{mv^2}{r} \quad \rightarrow \quad r = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{mv^2}
\]

But now, orbital angular momentum is quantized:

\[
L = mvr = n\hbar \quad \rightarrow \quad \frac{1}{v} = \frac{mr}{n\hbar}
\]

So the \textbf{n allowed} radii are:

\[
r_n = n^2 \frac{4\pi\varepsilon_0\hbar^2}{me^2} = n^2 a_0
\]

\[
n = 1, 2, 3, \ldots
\]

\[
r_1 = a_0 = 0.053 \text{ nm}, \quad r_2 = 4a_0 = 0.211 \text{ nm}, \quad r_3 = 9a_0 = 0.475 \text{ nm}
\]

Bohr radius
So, now let’s try to solve for the velocity.

Since

\[
\frac{1}{v} = \frac{mr}{n\hbar}
\]

\[
r_n = n^2 \frac{4\pi \epsilon_0 \hbar^2}{me^2} = n^2 a_0
\]

Then velocity is quantized, too.

\[
v_n = \frac{1}{4\pi \epsilon_0} \frac{e^2}{n\hbar} = \frac{\alpha c}{n} \approx \frac{1}{137} \frac{c}{n}
\]

\(\alpha\) is the called the “fine-structure constant”. Notice that it is dimensionless.

The fine-structure constant will show up in various places and has an important role in fundamental physics.
Ok, last part. Let’s solve for the energy. Since the velocity was not (quite) relativistic, we can perform a classical calculation.

\[ E = K + U = \frac{1}{2}mv^2 - \frac{1}{4\pi\epsilon_0} \frac{e^2}{r} \]

We can plug our previous expressions for the velocity and radius into this equation, as well. We now get a quantized energy.

\[ E_n = -\left( \frac{1}{4\pi\epsilon_0} \right)^2 \frac{me^4}{2n^2\hbar^2} = -\frac{mc^2\alpha^2}{2n^2} \]

This means that the **ground state** H atom is
So what does this mean?

$$E_n = -\frac{mc^2\alpha^2}{2n^2}$$

Energy is not only quantized, but it has a ground state \((n=1)\) below which the photon cannot fall.

$$E_1 \approx -(0.511 \text{ MeV}) \left( \frac{1}{137} \right)^2 \frac{1}{2} = 13.6 \text{ eV}$$

So we have shown that the Hydrogen atom is stable and it’s ground state is the measured ionization energy.

There are also excited states \((n>1)\). Let’s compare to the Rydberg equation.
The energy of a photon when a Hydrogen atom de-excite from one state to the next is

\[
E_\gamma = \frac{hc}{\lambda} = E_U - E_L = - \frac{mc^2 \alpha^2}{2} \left( \frac{1}{n_U^2} - \frac{1}{n_L^2} \right)
\]

Solving for the inverse wavelength, we get

\[
\frac{1}{\lambda} = - \frac{mc\alpha^2}{2h} \left( \frac{1}{n_U^2} - \frac{1}{n_L^2} \right) \approx -0.01097373 \text{ nm}^{-1} \left( \frac{1}{n_U^2} - \frac{1}{n_L^2} \right)
\]

But \( R_H = 0.01096776 \text{ nm}^{-1} \)

That’s really close, but can we do even better?
In fact the electron and the nucleus are revolve around a common mass (the nucleus is not infinitely massive, although, compared to the electron, it’s close).

Replace mass in the equation with the reduced mass:

\[ \mu = \frac{mM}{m + M} \]

where \( M \) is the mass of the nucleus. Once you do that, you get precisely the Rydberg constant \((M=M_{\text{proton}})\).

\[ R_H = -\frac{\mu c^2}{2\hbar} \approx 0.01096776 \text{ nm}^{-1} \]
To recap, from two assumptions, we have recovered the full atomic spectrum of hydrogen!
In the Bohr model, we have

\[ E_n = -\frac{13.6 \text{ eV}}{n^2} \quad \text{and} \quad r_n = a_0 n^2 \]

PRS Question: A electron in an Hydrogen atom drops from the n=6 state to the n=2 state. What is true about the relationship between \( E_6 \) and \( E_2 \)?

A) \( E_6 = E_2 / 9 \)
B) \( E_6 = E_2 / 4 \)
C) \( E_6 = E_2 \)
D) \( E_6 = 4E_2 \)
E) \( E_6 = 9E_2 \)
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\[
E_6 = -\frac{13.6 \text{ eV}}{6^2} \quad E_2 = -\frac{13.6 \text{ eV}}{2^2}
\]

\[
\frac{E_6}{E_2} = \frac{2^2}{6^2} = \frac{4}{36} = \frac{1}{9}
\]
Pre-Quantum era questions about the Hydrogen Atom
✓ Why does the Rydberg formula work?
✓ Why is the absorption the same as the emission spectrum?
✓ Why is the ionization energy of Hydrogen 13.6 eV?
✓ Why is the atom stable in the first place?

Addressed by the Bohr model. But the Bohr model is just that: a model. **Why should the angular momentum be quantized in the first place?** The answer will come with quantum mechanics.
Deficiencies in the Bohr Theory

• Many of the energy levels in the hydrogen are actually doublets, i.e. two levels closely spaced in energy (wavelength). Bohr’s theory cannot accommodate this.

• Quantization is simply assumed, but not derived.

• Spectra of more complicated elements (with more electrons) cannot be explained.

• Bohr theory is not relativistic. since $v \sim c/137$, it’s not so bad, but we know it can’t be a complete picture.
The discovery of deuterium (Urey, 1932)

Natural hydrogen is 1 part in 6000 deuterium (nucleus has one proton and one neutron, i.e. a deuteron).

So the reduced mass of deuterium is a little bit different than regular hydrogen.

\[
\mu_H = \frac{mM_H}{m + M_H} = 0.999456 \text{ } m
\]

\[
\mu_D = \frac{mM_D}{m + M_D} = 0.999728 \text{ } m
\]

The well known Balmer line (n=3 to n=2 transition) has wavelength 656.5 nm. But for deuterium, it is 656.3 nm!

In natural hydrogen, the regular emissions lines are accompanied by very faint lines from deuterium!