Relativistic Dynamics
Define relativistic momentum to be: \( \vec{p} = \gamma m \vec{v} \)

Notice how at velocities much less than the speed of light, relativistic momentum is the usual Newtonian momentum.

\[
\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2} \frac{v^2}{c^2} + \cdots \quad (v \ll c)
\]

(To get the above formula, perform a Taylor expansion about \( v/c \))

Defining momentum in this way is critical to preserving the concept of conservation of momentum.
Including the additional Lorentz factor $\gamma$ has a profound physical effect.

In Newtonian mechanics, doubling the velocity results in doubling the momentum (keeping mass fixed), and vice-versa.

In Relativistic mechanics, one cannot arbitrarily double the velocity of an object because doing so could result in exceeding the speed of light.
We can define the **relativistic force** in much the same way as we did in Classical mechanics:

\[ \vec{F} \equiv m\vec{a} = \frac{d\vec{p}}{dt} = \frac{d(\gamma mv)}{dt} \]

Suppose a particle starts at rest at \( x=0 \) and \( t=0 \), and experiences a **constant force** \( F \). What is the velocity as a function of time?

(Integrate both sides, remember \( F \) is a constant)

\[ \int F \, dt = \int dp \]

\[ Ft = p = \gamma mv = \frac{mv}{\sqrt{1 - v^2/c^2}} \]

(Solve for \( v \))

\[ v = \frac{Fct}{\sqrt{(Ft)^2 + (mc)^2}} \]
A particle experiencing a constant force will approach, but never exceed the speed of light.

\[ v = \frac{Ft}{m} = \frac{Fct}{\sqrt{(mc)^2}} \]

Classical

\[ v = \frac{Fct}{\sqrt{(Ft)^2 + (mc)^2}} \]

Relativistic
A note on the definition of mass in relativity.

Some textbooks will distinguish between rest mass and relativistic mass.

For instance, in the equation \( p = \gamma mv \):

the rest mass is “m”

the relativistic mass is “\( \gamma m \)”

The interpretation is that as the velocity of a particle increases, its (relativistic) mass also increases.

Although this is an interesting physical interpretation, it can also lead to confusion. We’ll follow T&R’s convention and refer to mass only as “rest mass.”

Our interpretation of mass will be as a relativistic invariant.
Just as we defined relativistic momentum, it makes sense to define a concept of energy $E$ for an object that fulfills the following criteria:

1. At low velocity, the relativistic energy matches the classical definition (up to a constant).
2. The total energy of an isolated system ($\Sigma E$) of bodies should be conserved in all inertial reference frames.

Recall that the work done by a force $F$ to move a particle from position 1 to 2 along a path $s$ is defined by

$$W_{12} = \int_{1}^{2} \vec{F} \cdot d\vec{s} = K_2 - K_1$$

...$K_1$ and $K_2$ being the particle’s kinetic energy (this is true, at least, for a frictionless system).
After some slightly “involved” algebra that you can follow in your textbook, we find that

\[ K = mc^2(\gamma - 1) \]

Notice that this is very different than the classical

\[ K = \frac{1}{2}mv^2 \]

But recall (from slide 2) that

\[ \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2} \frac{v^2}{c^2} + \cdots \quad (v \ll c) \]

And we see that we can recover the classical result...

\[ K = mc^2(\gamma - 1) = mc^2\left(1 + \frac{1}{2} \frac{v^2}{c^2} + \cdots - 1\right) \approx \frac{1}{2}mv^2 \]
So we have “derived” a notion of relativistic kinetic energy, but what about relativistic total energy?

Let’s rewrite the kinetic energy equation as:

\[ K = mc^2(\gamma - 1) \rightarrow \gamma mc^2 = K + mc^2 \]

It turns out that by defining the energy so that

\[ E = \gamma mc^2 = K + mc^2 \]

we recover conservation of energy (note that K, generally, is **not** conserved).

This is quite profound. It means that in the absence of any gravitational field, electromagnetic field, or event kinetic energy, a particle **still has a “rest energy”** proportional to its mass. (Where did this come from??)
A bit about units... The electron’s rest energy $E_0 = mc^2$ is

$= (9.11 \times 10^{-31} \text{ kg})(3.0 \times 10^8 \text{ m/s})^2 \approx 8.2 \times 10^{-14} \text{ J}$

It’s frequently more convenient to measure energy in units of electron volts:

The amount of energy gained by a single electron when moved across an electric potential difference of one volt.

1 eV = $1.6 \times 10^{-19} \text{ J}$, so $E_0 = 0.511 \text{ MeV}$

(equivalently, $m = 0.511 \text{ MeV/c}^2$)

This implies a new, useful unit of mass ($\text{MeV/c}^2$) and momentum ($\text{MeV/c}$)

Of course, we can also use eV/c, keV/c, GeV/c, etc...
Consider a pion ($m_{\pi}=140\text{ MeV/c}^2$) moving at $v=0.8c$. What is its **momentum**, **total energy**, and **kinetic energy**?

\[
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1 - 0.8^2}} \approx 1.67
\]

\[
p = \gamma mv = (1.67)(140 \text{ MeV/c}^2)(0.8 \text{ c}) = 187 \text{ MeV/c}
\]

\[
E = \gamma mc^2 = (1.67)(140 \text{ MeV/c}^2)c^2 = 233 \text{ MeV}
\]
There is a fundamental relationship between energy and momentum in relativistic mechanics:

\[ E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - \beta^2}} \quad (\beta \equiv \frac{v}{c}) \]

\[ p = \gamma mv = \frac{m\beta c}{\sqrt{1 - \beta^2}} \]

\[ E^2 - p^2 c^2 = \frac{m^2 c^4}{1 - \beta^2} - \frac{m^2 \beta^2 c^4}{1 - \beta^2} = \frac{m^2 c^4 (1 - \beta^2)}{1 - \beta^2} = m^2 c^4 \]

Therefore, we get the energy-momentum relation:

\[ E^2 = p^2 c^2 + m^2 c^4 \]
Let’s come back briefly to our original pion. It had mass of 140 MeV/c\(^2\) and it was moving at \(v=0.8c\).

We found that \(p=187\) MeV/c and \(E=233\) MeV. Let’s check the consistency with the energy-momentum relation:

\[
E^2=p^2c^2+m^2c^4=(187 \text{ MeV/c})^2c^2+(140 \text{ MeV/c}^2)^2c^4
\]
\[=
54569 \text{ MeV}^2
\]

taking square roots of both sides we get \(E=233\) MeV (same as above)

BTW, there is an easy way to solve for \(v\), given \(E\) and \(p\):

\[
\frac{p}{E} = \frac{\gamma mv}{\gamma mc^2} = \frac{v}{c^2}
\]

SO...

\[
v = \frac{pc^2}{E}
\]
What happens if $v=c$?

$$E = \gamma mc^2 = \frac{mc^2}{\sqrt{1 - v^2/c^2}}$$

If $v=c$, **then E is infinite!** Only way to avoid this is if the mass is 0, then $E=0/0$ (which can be finite).

This is exactly what happens with light. Photons are massless, so that $m=0$, $v=c$, and $E$ is finite. But what energy exactly?

Since $E^2=p^2c^2+m^2c^4$, **then $E=pc$ when $m=0$.**
Does light travel at the speed of light?

This is a little more nuanced than you might naively think. Although the velocity of light plays a special role in relativity, it’s real significance is that it exemplifies a universal velocity, whose value is the same in all frames.

We can slightly modify the question to read: “does light travel at the universal velocity?”

Theoretically, the photon is assumed to be massless (and so travels at the universal velocity) but there are experimental bounds on the mass, as well.

For instance...
Consider light from a pulsar in the Crab nebula (5000 ly away).

A pulsar is a rapidly rotating neutron star that emits electromagnetic radiation (see picture below).

![Pulsar](image)

(not an illustration)

The crab nebula pulsar is particularly interesting because it has a pulsing frequency of 30 Hz (!) and emits in optical, x-ray, and radio frequencies. Moreover, the pulse are quite sharp (widths of ~milliseconds).