POP QUIZ ON LENZ’S LAW

15 fine
3 mostly fine
8 need to review
Analyzing time-dependent circuits
• currents that change with time (capacitors and/or inductors in circuit)
• Emf sources whose emf changes with time

Described by linear differential equations

TODAY: AC circuits in steady state

Using complex numbers to represent the amplitude and phases of the voltages, currents, and capacitor charges

Single loop: differential equation ->linear equation
Multiloop circuit: Systems of differential equations -> systems of linear equations
Simple AC circuits: $\mathcal{E}(t) = E_m \cos(\omega_d t)$

“Steady state”: all currents oscillate at $\omega_d$; amplitudes don’t change with time

“driven”
Loop rule: \( \frac{Q(t)}{C} + Ri(t) + Li(t)/dt = E \cos(\omega t) \)
\( \frac{Q(t)}{C} + RdQ(t)/dt + L d^2Q(t)/dt^2 = E \cos(\omega t) \)

Steady state \( Q(t) = Q_A \cos(\omega t) + Q_B \sin(\omega t) \)

Substitute; solve separately for coefficients of \( \cos \) and \( \sin \)
Keep going to get \( I(t) \)
Simple AC circuits: \( E(t) = E_m \cos(\omega_d t) \)

\[ I(t) = \frac{E_0}{\sqrt{R^2 + (\omega L - 1/(\omega C))^2}} \cos(\omega t + \phi) \]

Maximum amplitude for \( I(t) \) when \( \omega^2 = 1/(LC) \) is RESONANCE.
Simple AC circuits: $\mathcal{E}(t) = E_m \cos(\omega_d t)$

driven RLC circuit

From p. 398: “The inductor and capacitor in series are equivalent to a single circuit element that is either an inductor or a capacitor, depending on whether $\omega^2/(LC)$ is greater or less than 1”

$$\omega L' = \omega L - \frac{1}{\omega C}$$
Key idea:
Represent an oscillating function $f(t) = f_m \cos(\omega t + \phi)$ by a complex number $f_t = f_m e^{i\phi}$
So $f(t) = \text{Re}(f_t e^{i\omega t})$

Incredibly useful facts:
The time derivative $df(t)/dt$ is represented by $i\omega f_t$
The time anti-derivative is represented by $f_t/(i\omega)$

You can check this yourself by showing
$\text{Re}(i\omega ft e^{i\omega t}) = df_t/dt$
and that
$d (\text{Re}((f_t/(i\omega)) e^{i\omega t}) /dt = f(t)$
Loop rule: \( \frac{Q(t)}{C} + Ri(t) + L \frac{di(t)}{dt} = E_m \cos(\omega t) \)

\( \frac{Q(t)}{C} + R \frac{dQ(t)}{dt} + L \frac{d^2Q(t)}{dt^2} = E_m \cos(\omega t) \)

Steady state \( Q(t) = Q_m \cos(\omega t + \phi) \)

= \text{Re}(Q_t e^{i\omega t})

Represent the charge by complex number \( Q_t = Q_m e^{i\phi} \)

\( I(t) = \frac{dQ(t)}{dt} = -\omega Q_m \sin(\omega t + \phi) \)

= \text{Re}(I_t e^{i\omega t})

Represent the current by \( I_t = i\omega Q_t \)

\( \frac{d^2Q(t)}{dt^2} = -\omega^2 Q_m \cos(\omega t + \phi) \)

Represented by \(-\omega^2 Q_t \)

\( \text{Emf } E_m \cos(\omega t) = \text{Re}(E_m e^{i\omega t}) \) so emf represented by \( E_m \)
Loop rule: \( \frac{Q(t)}{C} + R i(t) + L \frac{d}{dt} i(t) = E_m \cos(\omega t) \)

\( \frac{Q(t)}{C} + R dQ(t)/dt + L \frac{d^2}{dt^2} Q(t) = E_m \cos(\omega t) \)

\( (\frac{Q_t}{C} + R_i \omega Q_t - L \omega^2 Q_t) e^{i\omega t} = E e^{i\omega t} \)

Solve this linear equation for complex number \( Q_t \)

\( Q_t = \frac{E}{(1/C - \omega^2 L + i\omega R)} \)

\( Q(t) = \text{Re} (Q_t e^{i\omega t}) \)

\( I(t) = \text{Re}(i\omega Q_t e^{i\omega t}) \)

Get the amplitude and the phase
\[ E_m \cos \omega_d t - L \frac{di(t)}{dt} = 0 \]

\[ E_m - i \omega LI_t = 0 \]

\[ I_t = \frac{E_m}{i \omega L} \]

\[ i(t) = \text{Re}(I_t e^{i \omega t}) = \text{Re}((-i \frac{E_m}{\omega L})(\cos \omega t + i \sin \omega t)) \]

\[ i(t) = \left( \frac{E_m}{\omega L} \right) \sin \omega t \]
AC Network circuit

In each branch $I(t) = I_m \cos(\omega t + \phi)$
Represent by complex number $I_t = I_m e^{i\phi}$

**Junction rule**: the sum of the complex currents at a junction is zero
Why this works: math

The complex number representing the sum of two currents is the sum of the complex numbers representing the two currents

\[ I_{\text{tot}}(t) = I_m \cos(\omega t + \phi) \]
Loop rule

Change in complex voltage is obtained from complex current depending on the type of circuit element

\[ V_t = -R I_t \] resistor
\[ V_t = -\frac{1}{i\omega C} I_t \] capacitor
\[ V_t = -i\omega L I_t \] inductor

In each case, \( V_t \) is proportional to \( I_t \)
THESE SIGNS ARE CORRECT FOR PROPER APPLICATION OF THE LOOP RULE

Why are the signs opposite to the signs of the “impedances” of the circuit elements (see Table 8.1 in the book)??
Why is the sign opposite to the sign of the “impedance” of the circuit element (see Table 8.1 in the book)? It’s because of the way the impedance is defined!

Use these complex voltage changes in an AC circuit with one circuit element:
Resistor: $E - R I_t = 0$ so $E = R I_t$
Inductor: $E - L I_\omega I_t = 0$ so $E = L I_\omega I_t$
Capacitor: $E - (1/(i\omega C))I_t = 0$

The constants multiplying $I_t$ to get $E$ are called the impedances. In the book, $V_t$ is taken to be equal to $E$: therefore what the book calls $V_t$ is the complex voltage that you SUBTRACT as you go through the circuit element (the simplest example being that in the loop rule you SUBTRACT IR when you go through a resistor).
Analyzing AC circuits that can be reduced using series/parallel combinations
(here “\(V_t\)” is defined as in the book, which is the complex potential you subtract as you go through the circuit element)

“\(V_t\)” = \(Z I_t\)
\(Z\) is called the impedance
Two circuit elements in series can be replaced by one with an effective impedance that is the sum of the two impedances (same current)

\(I_t = Y \times V_t\)
\(Y\) is called the admittance
Two circuit elements in parallel can be replaced by one with an effective admittance that is the sum of the two admittances (same voltage)
Analyzing AC circuits that CANNOT be fully reduced using series/parallel combinations

EXAMPLE of such a circuit: Figure 8.14 in the book

Usually you should first look for combinations of circuit elements in series or in parallel and reduce the circuit as much as possible.

Then introduce complex currents to represent the oscillating current in each branch.
Use the junction rule and the loop rule to get a system of linear equations that allow you to solve for the complex currents.
Power and energy (see Section 8.6 in book for more details)

Instantaneous
\[ P(t) = I(t)V(t) \]

Average over a cycle
If \( I(t) \) and \( V(t) \) are in phase, then
\[ P = \frac{V_m I_m}{2} = V_{\text{rms}} I_{\text{rms}} \]

In general
\[ P_{\text{rms}} = V_{\text{rms}} I_{\text{rms}} \cos \phi \]
Where \( \phi \) is the phase difference between \( I \) and \( V \)