BEFORE CLASS: Pick up graded HW6
8th HW IS DUE IN CLASS TODAY
9th HOMEWORK ASSIGNMENT IS DUE IN CLASS NEXT MONDAY
EXAM #2 IS WED APRIL 11 IN CLASS

class web site
http://www.physics.rutgers.edu/ugrad/272
Ampere’s law

\[ \int_C \vec{B} \cdot d\vec{s} = \mu_0 \int_S \vec{J} \cdot d\vec{a} \]

\[ \int_C \vec{B} \cdot d\vec{s} = \int_S (\nabla \times \vec{B}) \cdot d\vec{a} \]

\[ \nabla \times \vec{B} = \mu_0 \vec{J} \]
Many different vector fields could have the same curl: 
\[ \nabla \times (F + \nabla f) = \nabla \times F \]
\[ \nabla \times \vec{B} = \mu_0 \vec{J} \] does not completely determine \( \vec{B} \).
\[ \nabla \times \vec{B} = \mu_0 \vec{J} \] does not completely determine \( \vec{B} \)

There is another property of \( \vec{B} \):

**All magnetic field lines form closed loops**

So the flux of \( \vec{B} \) through a closed surface \( S \) is **ZERO**

\[
\int_S \vec{B} \cdot d\vec{a} = 0 \quad \nabla \cdot \vec{B} = 0
\]

\[
\int_S \vec{B} \cdot d\vec{a} = \int_V (\nabla \cdot \vec{B}) dV = 0
\]
\[ \nabla \times \vec{B} = \mu_0 \vec{J} \]
\[ \nabla \cdot \vec{B} = 0 \]

completely determine \( \vec{B} \)

This fact is called the Helmholtz theorem

“A vector field is uniquely determined by its divergence and curl, assuming it goes to zero at infinity”
MAXWELL’S EQUATIONS
FOR TIME-INDEPENDENT CHARGE AND CURRENT DISTRIBUTIONS

\[ \nabla \times \vec{E} = 0 \quad \nabla \times \vec{B} = \mu_0 \vec{J} \]

\[ \nabla \cdot \vec{E} = \rho / \varepsilon_0 \quad \nabla \cdot \vec{B} = 0 \]

\( \rho \) determines \( E \) \quad \( J \) determines \( B \)
Verify $\text{div} \, \mathbf{B} = 0$ for infinite straight wire

In cylindrical coordinates

\[
\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{2\pi r} \hat{\theta}
\]

\[
\nabla \cdot \mathbf{B}(\mathbf{r}) = \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{\mu_0 I}{2\pi r} \right) = 0
\]
MAXWELL’S EQUATIONS
FOR TIME-INDEPENDENT CHARGE AND CURRENT DISTRIBUTIONS

\[ \nabla \times \vec{E} = 0 \quad \nabla \times \vec{B} = \mu_0 \vec{J} \]

\[ \nabla \cdot \vec{E} = \rho / \varepsilon_0 \quad \nabla \cdot \vec{B} = 0 \]

\( \rho \) determines \( \vec{E} \) \quad \( \vec{J} \) determines \( \vec{B} \)
Recall that for the electric field, there was an alternative description in terms of a function $\phi$ called the electric potential.

Key fact: the electric field is a conservative vector field.
Is the magnetic force a conservative force?

Is the magnetic field a conservative vector field?
The magnetic field is NOT a conservative vector field

Cross-derivative test

CANNOT write B as the gradient of a scalar function!
The magnetic field is NOT a conservative vector field

Cross-derivative test

CANNOT write B as the gradient of a scalar function!
B can be written as the curl of a vector field
A is called the “vector potential”

Find $A$ for the $B$ field of an infinite straight wire:

$\vec{A}(\vec{r}) = -\hat{z} \frac{\mu_0 I}{2\pi} \ln r$

Check by computing curl (see p. 294)

There is more than one choice of $A$

Many different vector fields could have the same curl:

$\nabla \times (\vec{F} + \nabla f) = \nabla \times \vec{F}$

$\nabla \times \vec{A} = \vec{B}$ does not completely determine $A$
Culture:
This freedom allows us to satisfy an extra condition on A: “gauge condition”

If the extra condition is div A = 0 ("Coulomb gauge")
Then $\nabla^2 \vec{A} = -\mu_0 \vec{J}$

$$\vec{A}(\vec{r}_1) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}_2)}{|\vec{r}_2 - \vec{r}_1|} dv_2$$

In Section 6.4 this is used to derive the Biot-Savart law – another relation between current distribution and magnetic field
Given any arrangement of current-carrying wires, find the magnetic field

First choice: use Ampere’s law if you can
Plan B: use Biot-Savart law
Calculating the Magnetic Field due to a Current: Biot-Savart Law

This element of current creates a magnetic field at $P$, into the page.

\[ d\vec{B} = \frac{\mu_0}{4\pi} \frac{i \, d\vec{s} \times \hat{r}}{r^2} \]  \hspace{1cm} \text{(Biot–Savart law)}.

\[ \mu_0 = 4\pi \times 10^{-7} \, \text{T} \cdot \text{m/A} \approx 1.26 \times 10^{-6} \, \text{T} \cdot \text{m/A}. \]

\[ \vec{B} = \int d\vec{B} \]
Magnetic Field due to a Long Straight Wire

This element of current creates a magnetic field at $P$, into the page.

$$dB = \frac{\mu_0}{4\pi} \frac{i \, ds \sin \theta}{r^2}.$$ 

$$B = 2 \int_0^\infty dB = \frac{\mu_0 i}{2\pi} \int_0^\infty \frac{\sin \theta \, ds}{r^2}.$$ 

$$r = \sqrt{s^2 + R^2}$$

$$\sin \theta = \sin(\pi - \theta) = \frac{R}{\sqrt{s^2 + R^2}}.$$ 

$$B = \frac{\mu_0 i}{2\pi} \int_0^\infty \frac{R \, ds}{(s^2 + R^2)^{3/2}}$$

$$= \frac{\mu_0 i}{2\pi R} \left[ \frac{s}{(s^2 + R^2)^{1/2}} \right]_0^\infty = \frac{\mu_0 i}{2\pi R}$$
Magnetic Field due to a Current in a Circular Arc of Wire

\[
d\vec{B} = \frac{\mu_0}{4\pi} \frac{i \, ds \times \hat{r}}{r^2} \quad \text{(Biot–Savart law).}
\]

\[
dB = \frac{\mu_0}{4\pi} \frac{i \, ds \sin 90^\circ}{R^2} = \frac{\mu_0}{4\pi} \frac{i \, ds}{R^2}.
\]

\[
B = \int dB = \frac{\mu_0 i}{4\pi R^2} \int ds = \frac{\mu_0 i}{4\pi R^2} R\phi
\]

\[
B = \frac{\mu_0 i \phi}{4\pi R} \quad \text{(at center of circular arc).}
\]

\[
B = \frac{\mu_0 i (2\pi)}{4\pi R} = \frac{\mu_0 i}{2R} \quad \text{(at center of full circle).}
\]

The right-hand rule reveals the field's direction at the center.
Magnetic Field on axis of circular ring

\[
\begin{align*}
d\vec{B} &= \dfrac{\mu_0}{4\pi} \dfrac{i \, ds \times \hat{r}}{r^2} \\
&= \dfrac{\mu_0 I b^2}{2(b^2 + z^2)^{3/2}} \\
\end{align*}
\]

(Biot–Savart law).

See Fig 6.15(a) and p. 299 in book

\[
B_z = \dfrac{\mu_0 I b^2}{2(b^2 + z^2)^{3/2}}
\]

When \(z=0\), this reduces to the field at the center of the ring, same as we found before
Magnetic Field on axis of straight cylindrical coil ("solenoid")

Current I, n turns/length, radius b

Consider each turn as a ring with current I at z
Compute the field on the axis at $z_0$

$$dB_z = \frac{\mu_0 b^2 I dz}{2(b^2 + (z_0 - z)^2)^{3/2}}$$

$$B_z = \int_{z_1}^{z_2} dB_z = \frac{\mu_0 b^2 I dz}{2(b^2 + (z_0 - z)^2)^{3/2}} = \frac{\mu_0 b^2 I}{2} \int_{z_1 - z_0}^{z_2 - z_0} \frac{dz'}{(b^2 + z'^2)^{3/2}}$$

$$= \frac{\mu_0 b^2 I n}{2} \left( \frac{z_2 - z_0}{b^2 (b^2 + (z_2 - z_0)^2)^{1/2}} - \frac{z_1 - z_0}{b^2 (b^2 + (z_1 - z_0)^2)^{1/2}} \right)$$

Limit for infinite coil is $\mu_0 I n$
Magnetic Field of rotating charged object

6.49 A disk with radius R and surface charge density $\sigma$ spins with angular frequency $\omega$.
What is the magnetic field at the center?

Spinning disk = set of rings of current
Ring at r with width dr has current $I(r)$

$$I(r) = \frac{\sigma 2\pi r dr}{2\pi / \omega} = \sigma \omega r dr$$

$$dB = \frac{\mu_0 I(r)}{2r} = \frac{\mu_0 \sigma \omega dr}{2}$$

$$B = \int_0^R dB = \frac{\mu_0 \sigma \omega R}{2}$$