TODAY

wave equation for E and B fields
Wave solutions of Maxwell’s equations
Energy and momentum in EM waves
Polarization of EM waves
Electromagnetism

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \]

Maxwell’s equations: fields, charge, current

\[ \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \]
Electric and magnetic fields in empty space

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \]

\[ \nabla \cdot \vec{E} = 0 \quad \nabla \cdot \vec{B} = 0 \]

Yes, E=B=0 is a solution BUT there are also solutions with nonzero E and B
Let’s use the vector identity

\[ \nabla \times (\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} ) \]

\[ \nabla (\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \]

\[ \nabla^2 \vec{E} - \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \]
\[ \frac{d^2 \vec{E}}{dx^2} + \frac{d^2 \vec{E}}{dy^2} + \frac{d^2 \vec{E}}{dz^2} - \mu_0 \varepsilon_0 \frac{d^2 \vec{E}}{dt^2} = 0 \]

\[ \frac{d^2 \vec{B}}{dx^2} + \frac{d^2 \vec{B}}{dy^2} + \frac{d^2 \vec{B}}{dz^2} - \mu_0 \varepsilon_0 \frac{d^2 \vec{B}}{dt^2} = 0 \]
Compare the wave equation for string

\[
\frac{d^2 \vec{E}}{dx^2} + \frac{d^2 \vec{E}}{dy^2} + \frac{d^2 \vec{E}}{dz^2} = \frac{1}{(\mu_0 \varepsilon_0)^{-1}} \frac{d^2 \vec{E}}{dt^2}
\]

\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}
\]

(wave equation).

\[
v = \sqrt{\frac{1}{\mu_0 \varepsilon_0}}
\]
Combine the Maxwell equations with \( \rho = j = 0 \) (empty space) to show
\[
\frac{d^2 \vec{E}}{dx^2} + \frac{d^2 \vec{E}}{dy^2} + \frac{d^2 \vec{E}}{dz^2} - \mu_0 \varepsilon_0 \frac{d^2 \vec{E}}{dt^2} = 0 \quad \frac{d^2 \vec{B}}{dx^2} + \frac{d^2 \vec{B}}{dy^2} + \frac{d^2 \vec{B}}{dz^2} - \mu_0 \varepsilon_0 \frac{d^2 \vec{B}}{dt^2} = 0
\]

compare
\[
\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \quad \text{(wave equation)}.
\]

E and B obey wave equations!
Wave speed = \((\mu_0 \varepsilon_0)^{-1/2} = 2.992 \times 10^8 \text{ m/s}
Compare measured speed of light
(in 1850, using mirrors 8 km apart)
c = 3.0 \times 10^8 \text{ m/s}
Maxwell’s aha moment!
Electromagnetic spectrum: Wavelength $\lambda = c/f$
One solution of these wave equations

\[\frac{d^2 \vec{E}}{dx^2} + \frac{d^2 \vec{E}}{dy^2} + \frac{d^2 \vec{E}}{dz^2} - \mu_0 \varepsilon_0 \frac{d^2 \vec{E}}{dt^2} = 0\]

\[\frac{d^2 \vec{B}}{dx^2} + \frac{d^2 \vec{B}}{dy^2} + \frac{d^2 \vec{B}}{dz^2} - \mu_0 \varepsilon_0 \frac{d^2 \vec{B}}{dt^2} = 0\]

\[E_y(x, y, z, t) = E_m \sin(kx - \omega t); \quad E_x = E_z = 0\]

\[B_x(x, y, z, t) = B_m \sin(k'x - \omega' t); \quad B_y = B_z = 0\]

With \(\omega/k = \omega'/k' = c\)
\[ E_y(x, y, z, t) = E_m \sin(kx - \omega t); \quad E_x = E_z = 0 \]

\[ B_x(x, y, z, t) = B_m \sin(k'x - \omega' t); \quad B_y = B_z = 0 \]

With \( \omega/k = \omega'/k' = c \)

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \nabla \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \]

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\[ \nabla \cdot \vec{E} = 0 \quad \nabla \cdot \vec{B} = 0 \]

NOT a solution of Maxwell’s equations!
Simplest example of fields that solve Maxwell’s equations: “Linearly polarized plane wave”

Must have \( \text{div } E = 0 \)
\( E_y(x, y, z, t) = E_m \sin(kx - \omega t) \)
\( E_x = E_z = 0 \)
\( \omega/k = c \)
Simplest example of fields that solve Maxwell’s equations: “Linearly polarized plane wave”

Must have $\text{div } E = 0$

$E_y(x, y, z, t) = E_m \sin(kx - \omega t)$

$E_x = E_z = 0$

$\frac{\omega}{k} = c$

then

$B_z(x, y, z, t) = \left(\frac{E_m}{c}\right)\sin(kx - \omega t)$
Electromagnetic wave traveling in $+x$ direction

E and B fields are perpendicular to direction of travel
E and B are perpendicular to each other
E/B = c

Changing B produces E
Changing E produces B
self-sustaining
Simple “linearly polarized” electromagnetic wave

Direction of propagation
Wavelength (or frequency)
Direction of the oscillating \( E \) field = “polarization”

\[
E_y(x, y, z, t) = E_m \sin(kx - \omega t)
\]

\[
B_z(x, y, z, t) = B_m \sin(kx - \omega t)
\]
Electromagnetic waves transport energy (radiation from the sun) measure using “intensity” = (energy/area)/time

$$I = \frac{\text{power}}{\text{area}} = \frac{P_s}{4\pi r^2},$$

where $P_s$ is the source power.
EM wave transports energy

Energy density associated with E field = \( (\varepsilon_0 / 2) |\vec{E}(\vec{r})|^2 \)

Energy density associated with B field = \( (1 / 2\mu_0) |\vec{B}(\vec{r})|^2 \)

\(?\frac{\partial}{\partial t} = \frac{\partial}{\partial t} E_x(x, y, z, t) = E_m \sin(kx - \omega t) \)

\( B_z(x, y, z, t) = B_m \sin(kx - \omega t) \)

Energy/area in one wavelength =
\[ \lambda \left( (\varepsilon_0 E_m^2 / 4) + (B_m^2 / (4\mu_0)) \right) \]
\[ = \lambda (\varepsilon_0 E_m^2 / 2) \]

(energy associated with E field = energy associated with B field)

Rate at which energy passes through unit area perpendicular to direction of propagation is (energy/area)/time
\[ c\varepsilon_0 E_m^2 / 2 = c\varepsilon_0 E_{rms}^2 \]
Rate at which energy passes through unit area perpendicular to direction of propagation is

\[ c \varepsilon_0 E_m^2 / 2 = c \varepsilon_0 E_{rms}^2 = I \]

“Energy current density” for a traveling wave

\[ \nabla \cdot \vec{S} = -\frac{\partial U(\vec{r},t)}{\partial t} \]

\[ \vec{S}(\vec{r},t) = (\vec{E} \times \vec{B}) / \mu_0 \]

Confirm it gives the right rate for a traveling wave (next slide)
33.5: Energy Transport and the Poynting Vector:

The direction of the Poynting vector $\vec{S}$ of an electromagnetic wave at any point gives the wave’s direction of travel and the direction of energy transport at that point.

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} \quad \text{(Poynting vector).}$$

Linearly polarized wave $E$ is perpendicular to $B$ and to direction of propagation $c \varepsilon_0 \varepsilon_0 E_m^2 = \text{speed} \times \text{energy per volume}$

$$S = \frac{1}{\mu_0} E B, \quad S = \frac{1}{c \mu_0} E^2$$

$$I = S_{avg} = \left(\frac{\text{energy/time}}{\text{area}}\right)_{avg} = \left(\frac{\text{power}}{\text{area}}\right)_{avg} = \frac{1}{c \mu_0} [E^2]_{avg} = \frac{1}{c \mu_0} [E_m^2 \sin^2(kx - \omega t)]_{avg}.$$
EM wave also transports **momentum**
(direction of momentum is parallel to direction of propagation)

Rate at which this momentum passes through unit area perpendicular to direction of propagation is \( \frac{I}{c} \)

\[
\text{(momentum/time)/area} = \frac{\text{force/area}}{\text{area}} = \text{pressure}
\]

“radiation pressure” \( P = \frac{I}{c} \)

If wave hits a perfectly reflecting surface, then \( P = 2\frac{I}{c} \)
Simple “linearly polarized” electromagnetic wave

Direction of propagation
Wavelength (or frequency)
Amplitude of the oscillating E field: intensity = power/area
Direction of the oscillating E field = “polarization”

\[ E_y(x, y, z, t) = E_m \sin(kx - \omega t) \]
\[ B_z(x, y, z, t) = B_m \sin(kx - \omega t) \]
“Polarizing sheet”

An electric field component parallel to the polarizing direction is passed *(transmitted)* by a polarizing sheet; a component perpendicular to it is absorbed.

transmitted  
absorbed  
???
An electric field component parallel to the polarizing direction is passed (transmitted) by a polarizing sheet; a component perpendicular to it is absorbed.

Component along polarizing direction is transmitted: $E \cos \phi$

Effect is to rotate the polarization and reduce the intensity $I$ proportional to $E_m^2$

So transmitted intensity $I = I_0 (\cos \phi)^2$

If the angle is 45 degrees, the intensity decreases by a factor of 2
“Unpolarized” light
Random mixture of directions
  e.g. light bulb

One polarizing sheet:
  Unpolarized light of intensity $I \rightarrow$ linearly polarized; $I = I_0/2$

Stack another polarizing sheet on top:
  Rotate: no difference ---- completely black
Two polarizing sheets are stacked with directions at right angles and placed on the projector, and transmit no light. A third sheet with direction at 45 degrees to each is inserted between the two sheets. What is then true about the intensity $I$ of the light transmitted by the 3-sheet stack?

a) No light is transmitted
b) $I = I_0$
c) $I = I_0/2$
d) $I = I_0/4$
e) $I = I_0/8$

ANSWER WITH THE DEMO!