Beads have normal force $N$ from ring.
By 3rd law the ring has an equal and opposite force

\[ mg \cos \theta - N = \frac{mv^2}{R} \]

\[ T - 2N \cos \theta - Mg = 0 \]

What is $V$ of bead?
Use energy.
\[ K_i = 0 \quad U_i = mgR \]
\[ K_f = \frac{1}{2}mv^2 \quad U_f = mgR \cos \theta \]

\[ \frac{1}{2}mv^2 + mgR \cos \theta = mgR \]

\[ \frac{V^2}{R} = 2g(1 - \cos \theta) \]
\[ mg \cos \theta - N = 2mg (1 - \cos \theta) \]

\[ N = 3mg \cos \theta - 2mg = mg (3\cos \theta - 2) \]

as \( T \to 0 \): \( 2N \cos \theta \max = -Mg \)

\[ 2mg (3\cos \theta \max - 2) \cos \theta \max = -Mg \]

\[ 6m \cos^2 \theta_m - 4m \cos \theta_m + M = 0 \]

\[ \cos^2 \theta_m - \frac{2}{3} \cos \theta + \frac{M}{6m} = 0 \]

\[ \cos \theta_m = \frac{1}{3} \pm \sqrt{\frac{1}{9} - \frac{M}{6m}} / 2 \]

\[ = \frac{1}{3} \pm \sqrt{\frac{1}{9} - \frac{M}{6m}} \]

\[ = \cos \theta_m \]

\[ \Rightarrow \text{when } m = \frac{3M}{2} \Rightarrow \cos \theta_m = \frac{1}{3} \text{ (term in } \sqrt{ } \text{ disappears)} \]

\[ \Rightarrow \text{if } m < \frac{3M}{2} \text{ then no solution for } \cos \theta_m \]

the ring will never rise.

\[ \Rightarrow \text{if } m \gg \frac{3M}{2} \text{ then one solution is } \cos \theta_m = \frac{2}{3} \]

it can never get bigger than that.

**Note:** If you assume opposite direction for \( N \), solution for \( N \) will have - sign. Everything will work out the same.
As discussed in recitation, there are two sources of 
\( F \) on the scale

1. weight of chain already fallen \( (2xg) \)

2. momentum flow chain hitting the scale now.

\[ dp = vdm = vL \, dx \]

\[ \frac{dp}{dt} = 2v^2 \]

\[ F_{\text{total on scale}} = 2v^2 + 2xg \]

But what is \( v \)? speed of top link of chain after it has fallen distance \( x = \) speed of chain link hitting scale

\[ v = \sqrt{2gx} \]

\[ F_{\text{total}} = 32gx \]
So \( F_{\text{Total}} = 3 \lambda g x \) until last link falls.

\[
F = \frac{3Mg x}{L}
\]

\( F_{\text{max}} = 3Mg \)

\( F_{\text{after}} = Mg \)

last link.

time of last link falling.
\[
\rho = \sqrt{a^2 + x^2}
\]

\[
U = \frac{-2GM_m}{\rho} = \frac{-2GM_m}{\sqrt{a^2 + x^2}}
\]

\[
K = \frac{1}{2} m v^2
\]

at \( x = 3a \):

\[
E = \frac{1}{2} m v_i^2 - \frac{2GM_m}{\sqrt{a^2 + 9a^2}} = \frac{1}{2} m v_i^2 - \frac{2}{\sqrt{10}} \frac{GM_m}{a}
\]

at \( x = 0 \):

\[
\frac{1}{2} m v_f^2 - \frac{2GM_m}{a}
\]
Solve for $U_f$

\[ \frac{1}{2} m U_f^2 = \frac{1}{2} m U_i^2 - \frac{2}{\sqrt{10}} \frac{GM}{a} + \frac{2GM}{a} \]

\[ U_f^2 = U_i^2 - \frac{4}{\sqrt{10}} \frac{GM}{a} + \frac{4GM}{a} \]

\[ U_f = \sqrt{U_i^2 + \left(4 - \frac{4}{\sqrt{10}}\right) \frac{GM}{a}} \]
\[ F = \frac{dp}{dt} = v \frac{dm}{dt} = v \frac{dm}{dt} \]

\( a) \)
Power:
\[ = \vec{F} \cdot \vec{v} = Fv = v^2 \frac{dm}{dt} \]

\( b) \)
\[ \text{dK of mass } dm \quad \frac{dm}{dt} \rightarrow v \]
\[ \text{dK} = \frac{1}{2} (\text{dm}) v^2 \]

\[ \frac{\text{dK}}{\text{dt}} = \frac{1}{2} \left( \frac{\text{dm}}{\text{dt}} \right) v^2 \]

which is half the power calculated in \( a) \).

the other half is going to dissipative force (ie friction) to get the sand moving at \( v \).
\[ \frac{dp}{dt} = F - mg \]

But, \[ \frac{dp}{dt} = v_0 \frac{dm}{dt} = v_0 \frac{(2dy)}{dt} = v_0 \frac{dy}{dt} = 2v_0^2 \]

\[ \frac{dp}{dt} = 2v_0^2 = F - mg = F - \lambda y g \]

\[ F = \lambda (v_0^2 + yg) \]
1) Power to nope:

\[ P_{\text{mech}} = F \cdot \vec{v} = F v_o = \lambda \left( v_o^2 + yg v_o \right) \]

\[ K = \frac{1}{2} m v_o^2 = \frac{1}{2} (\lambda y) v_o^2 \]

\[ U = \Theta \ m g h \quad \text{where} \ h = \frac{y}{2} \]

\[ U = m g \frac{y}{2} = \frac{1}{2} \lambda g y^2 \]

\[ E = \frac{1}{2} \lambda y v_o^2 + \frac{1}{2} \lambda g y^2 \]

Rate of change of \( E \):

\[ \frac{dE}{dt} = \frac{1}{2} \lambda v_o^2 \frac{dy}{dt} + \frac{1}{2} \lambda g y \frac{dy}{dt} \]

\[ \frac{dE}{dt} = \frac{1}{2} \lambda v_o^3 + \lambda g y v_o \]

\[ \frac{dE}{dt} = \lambda \left( \frac{v_o^3}{2} + yg v_o \right) \]

\( \lambda \) is smaller than Power

Energy is dissipated in getting rope moving.
$6.1$ Recall $5.13$

$$U = -\frac{2GMm}{\sqrt{a^2 + x^2}}$$

Expand around $x = 0$

$1^{st}$ deriv.
$$\frac{dU}{dx} = \frac{2GMm}{(a^2 + x^2)^{3/2}} x$$

$2^{nd}$ deriv.
$$\frac{d^2U}{dx^2} = \frac{2GMm}{(a^2 + x^2)^{3/2}} - \frac{6GMm}{(a^2 + x^2)^{5/2}} x^2$$

Taylor:
$$U(x) \approx U(0) + \frac{dU}{dx}\bigg|_{x=0} x + \frac{1}{2} \frac{d^2U}{dx^2}\bigg|_{x=0} x^2 + \cdots$$

$$U(x) = -\frac{2GMm}{\alpha} + O.x + \frac{1}{2} \left(\frac{2GMm}{(a^2 + X^2)^{3/2}}\right) x^2$$

Put it together:
$$U(x) \approx \{\text{const}\} + \frac{1}{2}\left[\frac{2GMm}{\alpha^3}\right] x^2 + \cdots$$
We can ignore the constant (and higher terms)

We compare to \( \Sigma + 0 \); \( U = \frac{1}{2} k \cdot x^2 \)

in this case: \( R = \frac{2GMm}{\alpha^3} \)

\[
\text{So } \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2GM}{\alpha^3}}
\]
6.3 Here’s the plan:

1. Write Newton’s 2nd law for each mass.
2. Use the symmetry of the problem to reduce 4 equations to 2.
3. Look for solutions in which all masses oscillate at the same frequency.
4. Solve the resulting equations.

\[ \begin{align*}
\ddot{x}_1 &= k(x_2 - x_1) \\
\ddot{x}_2 &= k(x_3 - x_2) - k(x_2 - x_1) \\
\ddot{x}_3 &= k(x_4 - x_3) - k(x_3 - x_2) \\
\ddot{x}_4 &= -k(x_4 - x_3)
\end{align*} \]

Only forces are the spring forces.

If we look for a solution with \( x_4 = x_1 \) and \( x_3 = x_2 \), the third and fourth equations will be automatically satisfied. When the first and second equations are satisfied (for example, if you replace \( x_1 \) by \( x_4 \) and \( x_2 \) by \( x_3 \) in the first equation, you get the 4th equation).

To get the solutions with \( x_4 = x_1 \) and \( x_3 = x_2 \), we need only to solve 2 equations for \( x_1 \) and \( x_2 \):

\[ \begin{align*}
\ddot{x}_1 &= k(x_2 - x_1) \\
\ddot{x}_2 &= k(x_2 - x_1)
\end{align*} \]
Write \( X_1(t) = c_1 \sin(wt + \phi) \) and \( X_2(t) = c_2 \sin(wt + \phi) \).

and substitute (cancel out the \( \sin(wt + \phi) \) factors)

\[-m\omega_1^2 c_1 = k(c_2 - c_1),\]

\[-m\omega_2^2 c_2 = k(c_1 - c_2).\]

so \( \omega_1^2(c_1 + c_2) = 0 \) (add the 2 equations)

either \( \omega^2 = 0 \) (all masses move together with same speed and relaxed springs)

or \( c_1 = -c_2 \) and \( \omega^2 = \frac{2k}{m} \)

(notice that the equations at the bottom of the previous page are the same as for the system discussed in class)

\[\rightarrow \text{Omm} \leftarrow \text{Omm} \leftarrow \text{snapshot of mode}\]

(notice we have 2 systems of two masses each oscillating out of phase so the spring in between is always relaxed)

REPEAT for other two solutions, as follows

Look for solutions with \( X_4 = -X_1 \) and \( X_3 = -X_2 \).

(Again, the 3rd & 4th equations will be satisfied if the 1st & 2nd are)

Solve these following two equations

\[mX''_1 = k(X_2 - X_1),\]

\[mX''_2 = k(X_1 - 3X_2).\]
Write $x_1(t) = c_1 \sin(wt + \theta)$ and $x_2(t) = c_2 \sin(wt + \phi)$.

and substitute

$\begin{align*}
-mw^2 c_1 &= k(c_2 - c_1) \\
-mw^2 c_2 &= k(c_1 - 3c_2)
\end{align*}$

$\Rightarrow c_1 = \frac{kc_2}{k-mw^2}$

$\Rightarrow -mw^2 c_2 = \frac{k^2 c_2}{k-mw^2} - 3k(k-mw^2)c_2$

if $c_2 = 0$, then $c_1 = 0$ — not a useful solution

then cancel our $c_2$

$-mw^2 = -2k^2 + 3kmw^2$

quadratic equation for $\beta = mw^2$

$\beta^2 - 4k\beta + 2k^2 = 0$

$mw^2 = \frac{4k \pm \sqrt{16k^2 - 4k^2}}{2}$

$w^2 = \left(2 \pm \frac{2}{m}\right)$

look at the displacement patterns:

$\begin{align*}
take c_2 &= 1, \quad c_1 &= \frac{k}{k-\left(2 \pm \frac{2}{m}\right)k} \\
&= \frac{1}{1 \pm \frac{2}{m}} = 1 \mp \frac{2}{m}
\end{align*}$

$W^2 = \left(2 \pm \frac{2}{m}\right)$

$W^2 = \left(2 - \frac{2}{m}\right)$
PS to 6.3 - Normal modes with linear algebra

Start with the 4 Newton's law equations

\[
\begin{align*}
mx_1' &= k(x_2 - x_1) \\
mx_2' &= k(x_1 - 2x_2 + x_3) \\
mx_3' &= k(x_2 - 2x_3 + x_4) \\
mx_4' &= -k(x_4 - x_3)
\end{align*}
\]

Look for solution \( x_i(t) = C_i \sin(wt + \phi) \)

The 4 equations for the \( 4 \) \( C_i \) can be arranged as follows:

\[
\begin{align*}
\left( \frac{k}{m} \right) (+a_1 - a_2) &= w^2 a_1 \\
\left( \frac{k}{m} \right) (-a_1 + 2a_2 - a_3) &= w^2 a_2 \\
\left( \frac{k}{m} \right) (-a_2 + 2a_3 - a_4) &= w^2 a_3 \\
\left( \frac{k}{m} \right) (-a_3 + a_4) &= w^2 a_4
\end{align*}
\]

This is an eigensystem for a 4x4 matrix

\[
\begin{pmatrix}
\frac{k}{m} & 1 & -1 & 0 \\
-1 & +2 & -1 & 0 \\
0 & -1 & +2 & -1 \\
0 & 0 & -1 & +1
\end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = w^2 \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}
\]

The 4 eigenvalues / eigenvectors of this matrix give you the 4 solutions.

Use symmetry (look for eigenvectors of form \( \begin{pmatrix} a \\ b \\ b \\ -a \end{pmatrix} \)) to get the explicit frequencies and displacement patterns.
\[ Q = 0 \]

\[ V = V' \]

\[ \text{elastic collision} \]

\[ mV_0 + 0 = MV - mV' \]

\[ V_0 = \frac{M}{m} \overline{V} - V' \]

**Energy cons.**

\[ \frac{1}{2} MV^2 + \frac{1}{2} mV'{}^2 = \frac{1}{2} mV_0^2 \]

\[ V_0^2 = \frac{M}{m} \overline{V}^2 + V'{}^2 \]

We need to eliminate \( \overline{V} \) since we do not know it (or care about it)

\[ V_0^2 - \frac{4}{9} V_0^2 = \frac{M}{m} V^2 \]

\[ V^2 = \frac{5m}{9M} V_0^2 \]

From Pcons: \[ V = \frac{5}{3} \frac{m}{M} V_0 \]
\[
\left(\frac{5}{3}\right)^2 \left(\frac{m}{M}\right)^2 \gamma_0^2 = \frac{5}{9} \frac{M}{m} \gamma_0^2
\]

\[
\frac{25}{9} \frac{m}{M} = \frac{5}{9}
\]

\[
M = 5m
\]
Prove in \( x \) & \( y \) separately:

\[ x: \; MV' \cos \theta = mV_0 - MV \]
\[ y: \; MV' \sin \theta - \frac{mV_0}{2} = 0 \]

Even more:

\[
\frac{1}{2} mV_0^2 + \frac{1}{2} MV^2 = \frac{1}{2} m \left( \frac{V_0}{2} \right)^2 + \frac{1}{2} MV^1^2
\]

we can eliminate \( V, V' \) & \( V_0 \)

\[ \cos \theta = \sin \theta = \frac{1}{\sqrt{2}} \]

\[ P_x \cos: \quad \frac{MV'}{\sqrt{2}} = mV_0 - MV \]

\[ P_y \cos: \quad \frac{MV'}{\sqrt{2}} - \frac{mV_0}{2} = 0 \quad \Rightarrow \quad V' = \frac{1}{\sqrt{2}} \frac{m}{M} V_0 \]
\[ \frac{\sqrt{E}}{\sqrt{M}} \left( \frac{1}{\sqrt{2}} \frac{m}{M} \sqrt{v_0} \right) = m v_0 - M V \]

\[ + \frac{1}{2} m v_0^2 = M V \]

\[ V = \frac{1}{2} \frac{m}{M} v_0 \]

**Euler:**

\[ \frac{1}{2} m \dot{v}_0^2 - \frac{1}{8} m \sqrt{v_0}^2 + \frac{1}{2} \frac{m}{M} \left( \frac{1}{2} \frac{m}{M} \sqrt{v_0} \right)^2 = \frac{1}{2} \frac{m}{M} \left( \frac{m}{M} \right)^2 \]

\[ = 0 \]

\[ \frac{1}{2} m \dot{v}_0^2 - \frac{1}{8} m \sqrt{v_0}^2 + \frac{1}{8} \frac{m^2}{M} \sqrt{v_0}^2 - \frac{1}{4} \frac{m^2}{M} \sqrt{v_0}^2 = 0 \]

\[ \frac{3}{8} \sqrt{v_0}^2 - \frac{1}{8} \frac{m}{M} \sqrt{v_0}^2 = 0 \]

\[ \frac{m}{M} = 3 \]
Translate to frame where wall is at rest (wall frame)

If wall at rest ($V = 0$)

$\Delta P = 2mV$

Time between collisions:

$v \Delta T = 2l \quad \text{distance by wall}$

$\Delta T = \frac{2l}{V}$

$F = \frac{\Delta P}{\Delta T} = \frac{2mV}{\frac{2l}{V}} = \sqrt{\frac{mV^2}{l}}$ \quad \text{time averaged force}
1) moving wall \((V \neq 0)\)

in wall frame: elastic bounce back to lab frame

\[ V' = (V+V) + V = V + 2V \]

change in \(V\)

\[ \Delta V = V' - V = 2V \]

time between collisions

\[ \Delta T = 2x \]

\[ \Delta T = \frac{2x}{V} \]

\[ \frac{\Delta V}{\Delta T} \Rightarrow \frac{dV}{dt} = \frac{2V}{2x/V} = \frac{VV}{x} \]

\[ \frac{dV}{dt} = \frac{dV}{dx} \frac{dx}{dt} \text{ (chain rule)} \]

\[ \frac{dV}{dx} = \frac{dV}{dt} \frac{1}{dx/dt} = -\frac{1}{V} \frac{dV}{dt} \]

\[ -V \text{ (since } x \text{ is wall separation, decreasing with } t) \]

\[ \frac{dV}{dx} = -\frac{1}{V} \left( \frac{VV}{x} \right) = -\frac{V}{x} \]
c) \( \frac{dv}{v} = -\frac{dx}{x} \) 

\[ \ln \left( \frac{v}{v_0} \right) = -\ln \left( \frac{x}{x_0} \right) \quad x_0 = l \]

\[ \frac{v}{v_0} = \frac{l}{x} \]

\[ v = \frac{v_0 l}{x} \]

Plug into \( F \) from part a) but with \( x \) as distance.

\[ F = \frac{Mv^2}{x} = \frac{Mv_0^2 l^2}{x^3} \]