Total mass \( M = \int_0^l 2 \, dx = \int_0^l A \cos \left( \frac{\pi x}{2l} \right) \, dx \)

\[ M = \frac{2A}{\pi} \sin \left( \frac{\pi x}{2l} \right) \bigg|_0^l = \left[ \frac{2A}{\pi} \right] = M \]

We can solve this for \( A = \frac{2M\pi}{l} \)

\[ \chi_{cm} = \frac{1}{M} \int_0^l x \, x \, dx = \frac{A}{M} \int_0^l x \cos \left( \frac{\pi x}{2l} \right) \, dx \]

Look up this integral

\[ \chi_{cm} = \frac{A}{M} \left[ \frac{4l^2}{\pi^2} \cos \left( \frac{\pi x}{2l} \right) + \frac{2lx}{\pi} \sin \left( \frac{\pi x}{2l} \right) \right] \bigg|_0^l \]

\[ \chi_{cm} = \frac{\pi}{2l} \left[ \frac{-4l^2}{\pi^2} + \frac{2l^2}{\pi} \right] = \frac{l}{\pi} \left( \frac{\pi}{2} - 2 \right) = \chi_{cm} \]
IF NO EXPLOSION
\[ \Rightarrow \text{PARABOLIC TRAJECTORY FOR ROCKET.} \]

WITH EXPLOSION AT PEAK, CM STILL GOES ON PARABOLIC TRAJECTORY
\[ \text{CM OF ROCKET HAS HORIZONTAL VELOCITY BUT NO VERTICAL VELOCITY AT EXPLOSION.} \]

\[ X_{cm} = L = \frac{M_A X_A + M_B X_B}{M_A + M_B} \]

GIVEN: \[ M_B = 3M_A \]
\[ X_A = -L \]
\[ X_{cm} = L \]
\[ \text{call } M_B = M, \ M_A = 3M \]
\[ L = \frac{M (-L) + 3M (X_B)}{4M} \]
\[ 5L = 3X_B \]
\[ X_B = \frac{5L}{3} \Rightarrow \frac{8L}{3} \text{ FROM LAUNCH SITE} \]
$M_1$ will lose contact with the wall when spring pulls on it. This will happen only when spring goes from compressed to stretched state.

THIS HAPPENS WHEN SPRING IS BACK TO LENGTH $L$
UNTIL M. LOSES CONTACT
TREAT $M_2$ MOTION AS SIMPLE HARMONIC MOTION
ABOUT $x_2 = l$

$x_2 = l + \left[ A \cos \omega t + B \sin \omega t \right] \quad \omega = \sqrt{\frac{k}{M_2}}$

$\dot{x}_2 = -A\omega \sin \omega t + B\omega \cos \omega t$

$\text{at } t = 0 \quad x_2 = \frac{l}{2} \quad (\text{initial condition})$

$\dot{x}_2 = 0$

$\begin{cases}
\left( \frac{l}{2} \right) = l + A(0) + B(0) & \Rightarrow A = 0, \quad -\frac{l}{2} \\
0 = -A\omega (0) + B\omega & \Rightarrow B = 0
\end{cases}$

$x_2 = l \quad \Rightarrow \frac{l}{2} \cos \omega t$ \quad until $x_2 = l$

When does $x_2 = l$?

$l = l - \frac{l}{2} \cos \omega t \quad \Rightarrow \text{when } \cos \omega t = 0 \quad \omega t = \frac{\pi}{2}$
\[
X_{\text{cm}}: \text{ FROM } t = 0 \text{ to } t = \frac{\pi}{2\omega} = \frac{\pi}{2\sqrt{k/m}}
\]

\[
X_{\text{cm}} = \frac{M_1 X_1 + M_2 X_2}{M_1 + M_2}
\]

Before \( t = \frac{\pi}{2\omega} \):
\[
X_1 = 0
\]
\[
X_2 = l - \frac{l}{2} \cos \omega t
\]

After \( t = \frac{\pi}{2\omega} \), \( M_1 \) loses contact with wall:
\[
X_1 = - \frac{A \omega \sin \omega t}{2} = \frac{l \omega \sin \left(\frac{\pi}{2}\right)}{2} = \frac{l \omega}{2}
\]

After \( t = \frac{\pi}{2\omega} \):
\[
X_2 = \frac{l \omega}{2} \text{ constant, and } X_1 = X_2 \text{ (no friction)}
\]

\[
X_{\text{cm}} \text{ \textit{after}} = \frac{l \omega}{2}
\]
We can use the rocket equation derived in lecture, BUT the external force is NOT zero.

\[
\frac{dP}{dt} = M \frac{dv}{dt} + u_0 \frac{dM}{dt}
\]

Since \( u_0 \) is negative

\[-uW = -uMg = M \frac{dv}{dt} + u_0 \frac{dM}{dt}\]

\[
\frac{dM}{dt} = \gamma (\text{given})
\]

\[
M(t) = M_0 - \gamma t
\]

\[-\gamma g(M_0 - \gamma t) = (M_0 - \gamma t) \frac{dv}{dt} + u_0 \gamma
\]

Solve for \( \frac{dv}{dt} \)

\[
\frac{dv}{dt} = \frac{u_0 \gamma}{(M_0 - \gamma t)} - \gamma g
\]
\[ V(t) =\int_0^t \left( \frac{1}{(M_0 - x(t))} \right) \, dx - ugt \]

\[ V(t) = -v_0 \left[ \ln (M_0 - x(t)) \right]_0^t - ugt \]

\[ V(t) = -v_0 \left[ \ln (M_0 - x(t)) - \ln (M_0) \right] - ugt \]

\[ V(t) = v_0 \left[ \ln \left( \frac{M_0}{M_0 - x(t)} \right) \right] - ugt \]

When is half the mass gone?

\[ M(t) = \frac{M_0}{2} = M_0 - x(t) \text{ half} \]

\[ t_{\text{half}} = \frac{M_0}{2v} \]

\[ V(t_{\text{half}}) = v_0 \left[ \ln \left( \frac{M_0}{M_0 - \frac{M_0}{2}} \right) \right] - ugt \frac{M_0}{2v} \]

\[ V(t_{\text{half}}) = v_0 \ln(2) - ugt \frac{M_0}{2v} \]

This is the maximum speed since after this it is slowing down.
Define
\[
\frac{dM}{dt} = -x
\]
\[
\frac{dt}{dM} = \frac{1}{-x}
\]

Assume \( U(t=0) = 0 \) (starts from rest)

Start with calculating \( P(t) \) & \( P(t+\Delta t) \)

Here \( M = M_{\text{car}} + M_{\text{sand}} \)

\[
P(t) = MV
\]

\[
P(t+\Delta t) = (M - \Delta M)(V + \Delta V) + \Delta m(V + \Delta V)
\]

Here \( \Delta m \) is the mass of sand that fell in \( \Delta t \)

\[
\Delta P = P(t+\Delta t) - P(t) = \left[ MV - \Delta MV + M\Delta V - \Delta mV\right] - MV
\]

\[
\Delta P = M\Delta V \quad \text{here } M \text{ is total mass}
\]

\[
\frac{dP}{dt} = M \frac{dV}{dt} = MA = F
\]

\[
\frac{dV}{dt} - \frac{F}{M} = \frac{F}{M_0 - xt}
\]
\[ \int_{0}^{t} dU' = F \int_{0}^{t} \frac{1}{M_0 - \delta t'} \, dt' \]

\[ U(t) = \left[ \frac{-\ln(M_0 - \delta t')}{\delta} \right]_{0}^{t} = -\frac{F}{\delta} \left[ \ln(M_0 - \delta t) - \ln(M_0) \right] \]

\[ U(t) = \frac{F}{\delta} \ln \left( \frac{M_0}{M_0 - \delta t} \right) \]

The question is what is the \( U(t = t_f) \) when all the sand is gone.

\[ \omega t = \frac{t_f}{t} \Rightarrow \delta t_f = m \text{ total mass of sand} \]

\[ U(t_f) = \frac{F}{\delta} \left( \ln \left( \frac{M_0}{M_0 - m} \right) \right) \]
6) 4.15

\[
\begin{align*}
\text{a) all } N \text{ men jump off at once:} \\
& \quad \text{Initial} = 0 \\
& \quad P_{\text{final}} = Mv_a + Nm(v_a - u) \\
& \quad \text{no external force:} \\
& \quad \text{Initial} = P_{\text{final}} \Rightarrow Mv_a = Nmv_a - Nmu \\
& \quad v_a = \frac{Nmu}{M + Nm}
\end{align*}
\]

\[
\begin{align*}
\text{b) jump off one at a time:} \\
\end{align*}
\]

1st jump:

\[
\begin{align*}
\text{Initial} = 0 \\
P_{\text{final}} = (M - m) v + m \left( \frac{mu}{Mm} - u \right) \\
= (M - m) (v + \frac{mu}{Mm}) + m (\frac{mu}{Mm} - u)
\end{align*}
\]
2nd jump

\[ P_i = \left( (N-1)m + M \right) U_i \]

\[ P_f = \left[ (N-2)m + M \right] U_2 + m \left( U_2 - U \right) \]

\[ (N-1)m + M \right] U_2 \Rightarrow mU = \left( (N-1)m + M \right) U_i \]

\[ U_2 = \frac{m}{(N-1)m + M} U_1 + U_i \]

Similarly

\[ U_3 = \frac{m}{(N-2)m + M} U_1 + U_2 \]

\[ U_{j+1} = \frac{m}{(N-2)m + M} U_j + U_{j} \]
\[ V_N = \sum_{m}^{N_m+M} \frac{m}{(N-1)m+M} \rightarrow \frac{m}{(N-2)m+M} \rightarrow \cdots \rightarrow \frac{m}{m+M} \frac{m}{3u} \]

call this \( V_6 \)

\[ V_a = \frac{N_m u}{N_m+M} = \sum_{m}^{N_m+M} \frac{m}{N_m+M} \rightarrow \frac{m}{N_m+M} \rightarrow \cdots \rightarrow \frac{m}{N_m+M} \frac{m}{3u} \]

In \( V_a \) all \( N \) terms have the same denominator.
In \( V_6 \), the 1st term has denominator \((N_m+M)\) but the rest of the terms have smaller denominators.

\( V_6 > V_a \)
Total length \( L \)
Total mass \( M \)

\[ F_{\text{ext}} = \frac{My}{L} g \]

where \( m \) is a function of time

\[ F_{\text{ext}} = MA \]

where \( a = \frac{d^2 y}{dt^2} \)

since no friction and rope does not stretch

\[ M \frac{d^2 y}{dt^2} = \frac{My}{L} g \]

\[ \frac{d^2 y}{dt^2} = \frac{g}{L} y \]

\( a) \) general solution

\[ y(t) = Ae^{\frac{g}{2L} t} + Be^{-\frac{g}{2L} t} \]

\[ \frac{d^2 y}{dt^2} = \left( \frac{g}{L} \right) y \]

\( b) \) at \( t=0 \)

\[ y = y_0, \quad \frac{dy}{dt} = 0 \]

\[ y(0) = A + B = y_0 \quad A = B = \frac{y_0}{2} \]

\[ \left. \frac{dy}{dt} \right|_{t=0} = \frac{g}{2L} \sqrt{\frac{g}{L}} (A - B) = 0 \]

\[ y(t) = \frac{y_0}{2} \left[ e^{\frac{g}{2L} t} + e^{-\frac{g}{2L} t} \right] \]
Force exerted on wall by stream of particles

Where \( \Delta t \) is the time interval

\[
\text{Force} = \frac{m}{\text{Unit Length}} \cdot \frac{\Delta (v' - v)}{\Delta t}
\]

where \( v' \) and \( v \) are the speeds of the incoming and outgoing streams of particles.

\[
\text{Velocity} = \sqrt{v^2 + w^2}
\]

\[
\text{Magnitude of change in velocity} = \frac{\Delta (v' - v)}{\Delta t}
\]
at height \( y \) water has speed \( V \):

\[
V = \sqrt{V_0^2 - 2gy}
\]

\[
\frac{dm}{dt} = K
\]

\[
dm = Kdt
\]

\[
dP_{up} = V \, dm = \sqrt{V_0^2 - 2gy} \, Kdt < \text{this is how much momentum}
\]

\[
dm \text{ of water imparts on impact with same speed}
\]

then momentum imparted to bucket is twice \( dP_{up} \) that at \( y_{\max} \) that \( W_{\max} \):  

\[
W = 2K \sqrt{V_0^2 - 2g y_{\max}}
\]
since the cone is $\theta = \frac{\pi}{2}$
The particles bounce off
1. to direction of motion
$\Rightarrow$ Reflected impact no moment
2. It all cancels out in $\sum$ all $\pm$ directions

So we only have to worry about incoming particle momentum

cross-section $A = \pi R^2$

How many particles? in time $\Delta t$

$\# \text{ of particles} = N A (\Delta t)$
each particle impacts $P_1 = m v$

so in time $\Delta t$:

$dP = (N A v \Delta t) m v = N A m v^2 \Delta t$
\[ F = \frac{dP}{dt} = -NAm \omega^2 = -b \omega^2 \]

where \( b = NAm = NTR^2m \)

\[ M \frac{dv}{dt} = -b \omega^2 \]

\[ \frac{1}{2} \int v^2 \, dt = \int \frac{b \omega}{M} \, dt \]

\[ v - v_0 = \frac{b \omega}{M} t \]

\[ \frac{1}{v^2} = \frac{b \omega}{M} t + \frac{1}{v_0} = \frac{b \omega v_0 + M}{Mv_0} \]

\[ v = \frac{Mv_0}{M + b \omega v_0} = \frac{v_0}{1 + \frac{b \omega v_0}{M}} \]