4.1 \[ M = \int \text{d}m = \int_0^l \text{d}x = A \int_0^l \cos \left( \frac{\pi x}{2l} \right) \text{d}x = \frac{2CA}{\pi} \]

\[ x = \frac{1}{M} \int_0^l x \text{d}m = \frac{1}{M} \int_0^l x \text{d}x = \frac{\pi}{2la} A \int_0^l x \cos \left( \frac{\pi x}{2l} \right) \text{d}x \]

\[ = l \left( 1 - \frac{2}{\pi} \right) \approx 0.36 l \]

4.2 Let \( \sigma \) be the area density of the plate. The total mass \( M \) is, then,

\[ M = \sigma \cdot \text{Area} = \sigma \frac{\sqrt{3}}{4} a^2 \]
It is easy to check that "Line 1" is given by the equation $y = \sqrt{3}x$.

Also, "Line 2" is $y = -\sqrt{3}x + \sqrt{3}a$.

For the center of mass,

$$X = \frac{1}{M} \iint \sigma x \, dx \, dy = \frac{\sigma}{M} \left( \int \int x \, dx \, dy + \int_{x=\frac{a}{2}}^{a} \int_{y=0}^{\sqrt{3}a} x \, dx \, dy \right) = \frac{4}{\sqrt{3}a^2} \left( \int_{x=0}^{\frac{a}{2}} x \cdot \sqrt{3}x \, dx + \int_{x=\frac{a}{2}}^{a} x \cdot (-\sqrt{3}x + \sqrt{3}a) \, dx \right) = \frac{a^2}{2}$$

Similarly,

$$Y = \frac{1}{M} \sigma \left( \int \int y \, dx \, dy + \int_{x=0}^{\frac{a}{2}} \int_{y=0}^{\sqrt{3}a} y \, dx \, dy \right) = \frac{4}{\sqrt{3}a^2} \left( \int_{y=0}^{\sqrt{3}a} \frac{3x^2}{2} \, dx + \int_{x=0}^{\frac{a}{2}} \frac{(\sqrt{3}a - \sqrt{3}x)^2}{2} \, dx \right) = \frac{a}{2\sqrt{3}}$$
The acrobat's velocity at the height $h$ is

$$v_h = \sqrt{v_0^2 - 2gh}$$

When he picks up the monkey, his velocity is reduced. According to the momentum conservation law,

$$P_{ini} = P_{final}, \quad Mv_h = (M+m)v_{pair},$$

where $v_{pair}$ is the velocity of the system "acrobat + monkey" after the monkey is picked up. The maximum height above the perch that the pair attains is reached when $v_{pair}(t) = 0$.

$$v_{pair}(t) = v_{pair}(0) - gt \quad \Rightarrow \quad t_{\max} = \frac{v_{pair}(0)}{g} \quad \Rightarrow \quad \text{the max height above the perch}, \quad h_{\max} = v_{pair}(0)t_{\max} - \frac{gt_{\max}^2}{2} =$$

$$= \frac{v_{pair}(0)^2}{2g} = \frac{1}{2g} \left( \frac{M}{M+m} \right)^2 v_h^2 = \frac{1}{2g} \left( \frac{M}{M+m} \right)^2 (v_0^2 - 2gh)$$

$$H_{\max} = h + h_{\max} = h + \frac{1}{2g} \left( \frac{M}{M+m} \right)^2 (v_0^2 - 2gh) =$$

$$= h \left(1 - \left(\frac{M}{M+m}\right)^2\right) + \frac{v_0^2}{2g} \left(\frac{M}{M+m}\right)^2.$$
Let \( M \) be the mass of the plane, \( m \) - mass of the sandbag, \( V_0 \) - speed of the plane at landing, \( \mu \) - friction coefficient, \( f_\text{add} \) - additional retarding force from the brakes. According to momentum conservation law, the speed of the "plane + bag" system, \( V_{ps} \) is initial given by

\[
MV_0 = (M+m)V_{ps}^0
\]

The total retarding force acting on the "plane + bag" is

\[
f_r = \mu mg + f_\text{add} = \text{const.}
\]

It results in a constant retarding acceleration

\[
a = \frac{f_r}{M+m} = \frac{\mu mg + f_\text{add}}{M+m}
\]

The plane will stop after time \( t_{stop} \) given by the condition

\[
0 = V(t) = V_{ps}^0 - at_{stop}
\]

The distance \( d \) that the plane will travel before it stops is

\[
d = V_{ps}t_{stop} - \frac{a t_{stop}^2}{2} = \left(\frac{V_{ps}^0}{a}\right)^2 \left(\frac{M}{M+m}\right)^2 \frac{V_0^2}{2} \frac{1}{\mu mg + f_\text{add}} = \frac{M^2}{M+m} \frac{V_0^2}{2} \frac{1}{\mu mg + f_\text{add}} = 12.72 \text{ ft}
\]
4.8 The initial velocity of the woman is \( v_0 = \sqrt{2g h} \).

Impulse = \( \int F \, dt = P_{\text{final}} - P_{\text{initial}} = M v_0 - 0 = M \sqrt{2g h} = 193 \, \text{kg} \cdot \text{m/s} \)

4.10 The relative velocity \( \vec{u} \) of the sand leaving the car is zero! Therefore, according to the rocket equation, \( \vec{F}_{\text{ext}} = M(t) \frac{d\vec{u}}{dt} = \vec{u} \frac{dM(t)}{dt} = M(t) \frac{d\vec{u}}{dt} \).
(And even if the vertical component of \( \vec{u} \) is not zero, this is still correct for the horizontal direction, as long as \( u_x = 0 \).)

For the horizontal direction, we have

\[ F = M(t) \frac{d\vec{u}}{dt} \]

To avoid confusion, let me designate \( R = \frac{dm}{dt} = \text{const.} \)

Then, \( M(t) = M + m - R t \)

\[ \frac{d\vec{u}}{dt} = \frac{F}{M(t)} = \frac{F}{M + m - R t} \]

\[ \vec{u} = \int \frac{F \, dt}{M + m - R t} = \frac{F}{R} \ln \left( \frac{M + m}{M + m - R t} \right) \]

When all the sand is gone, \( R t = m \),

\[ \vec{u} = \frac{F}{dm/dt} \ln \left( \frac{M + m}{M} \right) \]
a) Momentum conservation: $0 = M \dot{V} + \sum_{i=1}^{N-n} m_i (V_n - u_i)$

$v_n = \frac{m \dot{V}}{M + N m}$

b) Let $v_n$ be the velocity of the platform after $n$ men jumped off. Momentum conservation:

$(M + [N-(n-1)]m) v_{n-1} = (M + [N-n]m) v_n + m \left( v_n - u \right)$

$v_n = v_{n-1} + \frac{m v_n}{M + (N-n+1)m}$

$v_{\text{final}} = v_N = \sum_{i=1}^{N} \frac{m}{M + (N-i+1)m}$

c) $v_a = u m \sum_{i=1}^{N} \frac{i}{M + N m}$

$v_b = u m \sum_{i=1}^{N} \frac{1}{M + (N-i+1)m}$

Obviously, $v_b > v_a$

It is easy to see that $v_a < u$, $v_{a,\text{max}} \to u$ (why?)

For small $m$ and large $N$, $v_b \to u \ln \frac{M + m N}{M} = u \ln \frac{\text{Mini}}{\text{Max}}$

$v_b$ can be arbitrarily large. The limiting expression for $v_b$ is the same as for a rocket in free space, as we would have expected.
The force acting on the rope is

\[ F_x = M \frac{x(t)}{\ell} g \]

We can think about the motion of the rope as of a one-dimensional motion with the \(x\)-dependent force \(F_x\). Applying this force to the center of mass,

\[ M \ddot{x} = F_x = M \frac{x}{\ell} g \]

\[ \ddot{x} - \frac{g}{\ell} x = 0 \]

\[ x(t) = A e^{\lambda t} + B e^{-\lambda t} \]

\[ \lambda = \sqrt{\frac{g}{\ell}} \]

Since \( x(0) = A + B = \ell_0 \)

and \( x'(0) = \lambda A - \lambda B = 0 \)

\[ A = B = \frac{\ell_0}{2} \]

\[ x(t) = \frac{\ell_0}{2} (e^{\lambda t} + e^{-\lambda t}) = \ell_0 \cosh \left( \sqrt{\frac{g}{\ell}} t \right) \]