Midterm 1: Monday October 5th 2014

- Motion in one, two and three dimensions
- Forces and Motion I (no friction)
- No energy and work.
6. Forces and Motion II

- Dynamics of uniform circular motion

Centripetal acceleration:

\[ \vec{a} = -\frac{v^2}{r} \hat{r} \]

- \( r \) = radius of the circle
- \( v \) = speed
- \( \hat{r} = \frac{\vec{r}}{r} \) unit radial vector
Newton’s 2nd law:

\[ \vec{F}_{\text{net}} = m\vec{a} \]

\[ \Downarrow \]

\[ \vec{F}_{\text{net}} = -\frac{mv^2}{r}\hat{r} \]

**Centripetal Force**

A centripetal force accelerates a body by changing the direction of the body’s velocity without changing the body’s speed.
• ball attached to a string in uniform circular motion on a horizontal frictionless plane

\[ \vec{F}_{\text{net}} = \vec{T} \text{ (tension)} \]

• the string breaks ⇒ uniform linear motion, according to Newton’s 1st law.

• no \( \vec{T} \) ⇒ no centripetal force ⇒ \( \vec{a} = 0 \)
i-Clicker

A ball attached to a string rotates in a vertical plane near Earth’s surface such that the string is stretched taut all points on the circular path. What is the minimum value of its speed at the highest point A of the trajectory.

\[ A) \ v_{\text{min}} = 0 \]
\[ B) \ v_{\text{min}} = \sqrt{rg} \]
\[ C) \ v_{\text{min}} = \sqrt{rg/2} \]
\[ D) \ v_{\text{min}} = \sqrt{2gr} \]
\[ E) \ \text{none of the above} \]
Answer

A ball attached to a string rotates in a vertical plane near Earth’s surface. For a point \( P \) on the trajectory, let \( T_P \) denote the magnitude of the tension in the string. Which of the following statements is true?

\[
\begin{align*}
A) \quad & v_{\text{min}} = 0 \\
B) \quad & v_{\text{min}} = \sqrt{rg} \\
C) \quad & v_{\text{min}} = \sqrt{rg/2} \\
D) \quad & v_{\text{min}} = \sqrt{2gr} \\
E) \quad & \text{none of the above}
\end{align*}
\]
\[ \vec{F}_{net} = -\frac{mv^2}{r} \hat{r} \quad \vec{F}_{net} = \vec{F}_g + \vec{T} \]

\[ (F_{net})_y = -(mg + T) = -\frac{mv^2}{r} \]

\[ T = \frac{mv^2}{r} - mg \geq 0 \]

\[ v^2 \geq rg \quad \Rightarrow \quad v_{min} = \sqrt{rg} \]
• **Example:** A ball attached to a string of length $l$, which makes an angle $\theta$ with the vertical, rotates uniformly in a horizontal plane as shown below. Find the speed $v$. 

![Diagram of a ball on a string](image)

- $l$: length of the string
- $\theta$: angle with the vertical
- $r$: radius of the circular path

The forces acting on the ball are:
- $T_y$: vertical component of the tension force
- $T_x$: horizontal component of the tension force
- $F_g$: gravitational force

The tension force $T$ can be resolved into its components $T_y$ and $T_x$.

The centripetal force $F_c$ is given by $F_c = \frac{m v^2}{r}$, where $m$ is the mass of the ball and $r$ is the radius of the circular path.

By resolving forces in the horizontal direction, we get $T_x = m \frac{v^2}{r}$.

By resolving forces in the vertical direction, we get $T_y - F_g = 0$.

Solving these equations gives the speed $v$. 

$$v = \sqrt{\frac{F_c r}{m}}$$
\[ \vec{F}_{\text{net}} = m\vec{a} \quad \vec{F}_{\text{net}} = \vec{F}_g + \vec{T} \]

\[ (F_{\text{net}})_x = T\sin \theta \]
\[ (F_{\text{net}})_y = -mg + T\cos \theta \]

\[ a_x = \frac{v^2}{r} \quad a_y = 0 \]

\[ T\sin \theta = \frac{mv^2}{r} = \frac{mv^2}{l\sin \theta} \]
\[ T\cos \theta - mg = 0 \]

\[ \frac{v^2}{l\sin \theta} = \tan \theta \quad \Rightarrow \quad v = \sin \theta \sqrt{\frac{lg}{\cos \theta}} \]
**Example:** a car of mass $m$ moving with constant speed makes a turn of radius $r$. Suppose the static friction coefficient between the wheels and the road is $\mu_s$. Find the maximum value of the speed $v$ such that the car will remain on the road.
\[ \vec{F}_{\text{net}} = m\vec{a} \quad \vec{a} = -\frac{mv^2}{r}\hat{i} \]
\[ F_N - mg = ma_y = 0 \]
\[ -f_s = ma_x = -\frac{mv^2}{r} \]
\[ f_s \leq \mu_s F_N \quad \Rightarrow \quad \frac{mv^2}{r} \leq \mu_s mg \]
\[ v \leq \sqrt{\mu_s rg} \]
7. Kinetic energy and work

- **What is energy?**
  - **Basic Idea:** scalar quantity with quantifies the state of motion and/or the capacity for motion of a system
  - **Conserved** as the state of motion of the system changes.
Speed of train at the lowest point?

Speed of arrow in flight?
Energy

- **Kinetic energy**: associated to motion
- **Potential energy**: associated to the capacity of generating motion

**Example**: freely falling ball from height $h$.

- Initial speed: $v_0 = 0$
- Final speed:
  \[ v^2 = v_0^2 + 2gh = 2gh \]
• initially: **potential energy**: $mgh$

• finally: **kinetic energy**: $mv^2/2$

• **Energy conservation**:

\[ mgh = \frac{mv^2}{2} \]
• **Kinetic Energy:** energy associated to the motion of an object

\[ K = \frac{1}{2}mv^2 \]

**Unit:** Joule

\[ 1\text{J} = 1\text{kg} \times \text{m}^2/\text{s}^2 \]
Non-zero force

⇓

Acceleration: change in velocity

⇓

Change in kinetic energy

How does $K$ change under an applied force?
• **Work**: energy transferred to or from an object by means of a force acting on the object.

Energy transferred *to* the object is **positive work** while energy transferred *from* the object is **negative work**.
How do we calculate the work of a force?

**Example:** bead on frictionless rod subject to constant force \( \vec{F} \)

Newton’s 2nd law:

\[
F_x = ma_x
\]

Constant acceleration model:

\[
v_x^2 = v_0^2 + 2a_x \Delta x
\]

\[
\frac{mv^2}{2} - \frac{mv_0^2}{2} = F_x \Delta x
\]
Work done by the force $\vec{F}$

\[ F \cos \phi = \vec{F} \cdot \vec{d} \]

\[ \vec{d} = (\Delta x)\hat{i} \]

displacement vector

\[ F \cos \phi \begin{cases} > 0, & \text{for } 0 \leq \phi < \pi/2 \text{ positive work, } K_f > K_i \\ = 0, & \text{for } \phi = \pi/2 \text{ zero work, } K_f = K_i \\ < 0, & \text{for } \pi/2 < \phi \leq \pi \text{ negative work, } K_f < K_i \end{cases} \]
**Net work:**

When two or more forces act on an object, the net work done on the object is the sum of the works done by the individual forces.

\[
W_{\text{net}} = \sum W = \sum \vec{F} \cdot \vec{d} = (\sum \vec{F}) \cdot \vec{d} = \vec{F}_{\text{net}} \cdot \vec{d}
\]
• **Work-Kinetic Energy Theorem**

\[
\text{(change in the kinetic energy of a particle)} = \text{(net work done on the particle)}.
\]

\[
\Delta K = W \quad K_f = K_i + W
\]
An object of mass $m = 1\, \text{kg}$ is launched with initial speed $v_0 = 2\, \text{m/s}$ along a rough horizontal surface. What is the total work done by the frictional force until it stops?

\[
\begin{align*}
A) & \quad 2\, \text{J} \\
B) & \quad 1\, \text{J} \\
C) & \quad -1\, \text{J} \\
D) & \quad -2\, \text{J} \\
E) & \quad 0\, \text{J}.
\end{align*}
\]
Answer

An object of mass $m = 1\text{ kg}$ is launched with initial speed $v_0 = 2\text{ m/s}$ along a rough horizontal surface. What is the total work done by the frictional force until it stops?

\[ W = \Delta K = K_f - K_i = -\frac{mv_0^2}{2} \]

A) 2 J
B) 1 J
C) −1 J
D) −2 J
E) 0 J.
A ball of mass $m$ attached to string is launched with initial speed $v_0$ on a circular trajectory on horizontal plane. The friction force between the ball and the surface has constant magnitude $f_k$. What is the speed of the ball after travelling a distance $s$ along the circle.

\[
\begin{align*}
A) \quad v &= v_0 \\
B) \quad v &= \sqrt{v_0^2 - \frac{2f_k s}{m}} \\
C) \quad v &= v_0 - \frac{f_k s}{mv_0} \\
D) \quad \text{cannot be determined from the data.}
\end{align*}
\]
Answer

A ball of mass $m$ attached to string is launched with initial speed $v_0$ on a circular trajectory on horizontal plane. The friction force between the ball and the surface has constant magnitude $f_k$. What is the speed of the ball after travelling a distance $s$ along the circle.

A) $v = v_0$

B) $v = \sqrt{v_0^2 - \frac{2f_k s}{m}}$

C) $v = v_0 - \frac{f_k s}{mv_0}$

D) cannot be determined from the data.
Infinitesimal displacement
\[ \vec{d}s = \vec{v}dt \]

Note that \( T \perp \vec{d}s \) while \( \vec{f}_k \) makes an angle \( \theta = 180^\circ \) with \( \vec{d}s \).

Infinitesimal work:
\[ dW_{f_k} = \vec{f}_k \cdot \vec{d}s = -f_k ds \]

Total work:
\[ W = \int dW = -f_k s \]

Work-energy theorem:
\[ \frac{mv^2}{2} - \frac{mv_0^2}{2} = -f_k s \]
\[ v = \sqrt{v_0^2 - \frac{2f_k s}{m}} \]
Work done by the gravitational force

Freely falling object moving downwards:

\[ W = \vec{F}_g \cdot \vec{d} = mgd \cos 0^\circ = mgd > 0 \]

\[ K_1 = \frac{mv_1^2}{2} \quad K_2 = \frac{mv_2^2}{2} \]

\[ K_2 - K_1 = mgd > 0 \]
Freely falling object moving upwards:

\[ W = \vec{F}_g \cdot \vec{d} = mgd \cos 180^\circ = -mgd > 0 \]

\[ K_1 = \frac{mv_1^2}{2} \quad K_2 = \frac{mv_2^2}{2} \]

\[ K_2 - K_1 = -mgd < 0 \]
An object of mass $m$ is launched with initial speed $v_0$ along an inclined plane making an angle $\theta = 45^\circ$ with the horizontal. The kinetic friction coefficient between the object and the plane is $\mu_k = 0.5$. Let $W_{f_k}$ be the total work done by the friction force until it stops. Which of the following statements is false?

A) $W_{f_k} < 0$

B) $W_{f_k} = -mv_0^2/2$

C) $|W_{f_k}| < mv_0^2/2$
**Answer**

An object of mass $m$ is launched with initial speed $v_0$ along an inclined plane making an angle $\theta = 45^\circ$ with the horizontal. The kinetic friction coefficient between the object and the plane is $\mu_k = 0.5$. Let $W_{fk}$ be the total work done by the friction force until it stops. Which of the following statements is false?

- **A)** $W_{fk} < 0$
- **B)** $W_{fk} = -mv_0^2/2$
- **C)** $|W_{fk}| < mv_0^2/2$
\[ \vec{F}_{\text{net}} = m\vec{a} \quad (F_{\text{net}})_x = ma_x \quad (F_{\text{net}})_y = ma_y = 0 \]

\[ \vec{F}_{\text{net}} = \vec{F}_g + \vec{F}_N + \vec{f}_k \]

\[ (F_{\text{net}})_x = -mg\sin\theta - f_k \quad (F_{\text{net}})_y = F_N - mg\cos\theta = 0 \]

\[ F_N = mg\cos\theta \quad f_k = \mu_k mg\cos\theta \]
\[ f_k = \mu_k mg \cos \theta \]

\[ h = ds \sin \theta \]

Work done by kinetic friction:
\[ W_{f_k} = \vec{f_k} \cdot \vec{d} = -f_k d = -\mu_k mg d \cos \theta \]

Work done by gravitational force:
\[ W_{F_g} = \vec{F_g} \cdot \vec{d} = -mgh = -mgd \sin \theta \]

Work done by normal force
\[ \vec{F_N} \cdot \vec{d} = 0 \]
Total work:
\[ W = W_{fk} + W_{Fg} = -mgd\left(\sin \theta + \mu_k \cos \theta\right) \]

Work-Kinetic energy theorem:
\[ W = \Delta K = 0 - K_i = -\frac{mv_0^2}{2} \]

Work done by kinetic friction:
\[ W_{fk} = \frac{\mu_k \cos \theta}{\frac{\sin \theta + \mu_k \cos \theta}{3}} = \frac{1}{3} \]
\[ W_{fk} = -\frac{mv_0^2}{6} \]