Final Exam:

- Wednesday, Dec 21st, 8:00-11:00am in PHL.
- Final make up: Thursday, Dec 22nd, 10:00am - 1:00pm in Serrin (physics building) E372.
- Final: 20 questions = 25% final score
13. Gravitation

Newton’s law of gravitation

Every point particle attracts every other particle with a gravitational force

\[ F = G \frac{m_1 m_2}{r^2} \]

\[ G = 6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2 = 6.67 \times 10^{-11} \text{ m}^3/\text{kg} \cdot \text{s}^2 \]
Gravitational force on particle 1 – vector form:

\[ \vec{F}_{2 \text{ on } 1} = G \frac{m_1 m_2}{r^2} \hat{r} \]

\( \hat{r} \) is the radial **unit** vector: \( |\hat{r}| = 1 \).
• **Principle of Superposition**

Given $n$ interacting particles:

\[ \vec{F}_{1,\text{net}} = \vec{F}_{12} + \vec{F}_{13} + \cdots + \vec{F}_{1n} \]

\[ \vec{F}_{1,\text{net}} = \text{net gravitational force acting on particle 1} \]

\[ \vec{F}_{1,i} = \text{gravitational force on particle 1 from particle } i \]
Example

We want the forces (pulls) on particle 1, *not* the forces on the other particles.

This is the force (pull) on particle 1 due to particle 2.

This is the force (pull) on particle 1 due to particle 3.

- Isolated system of particles far from other massive objects.
- What is the magnitude of the gravitational force on particle 1?

\[
\vec{F}_{1, \text{net}} = \vec{F}_{12} + \vec{F}_{13}
\]
\[ \vec{F}_{12} = G \frac{m_1 m_2}{a^2} \hat{j} \]

\[ \vec{F}_{13} = -G \frac{m_1 m_3}{4a^2} \hat{i} \]

\[ \vec{F}_{1,\text{net}} = \frac{G m_1}{a^2} \left( m_2 \hat{j} - \frac{m_3}{4} \hat{i} \right) \]

\[ |\vec{F}_{1,\text{net}}| = \frac{G m_1}{a^2} \sqrt{m_2^2 + \frac{m_3^2}{16}} \]
Gravitational force on a particle from an extended object:

\[ \vec{F} = \int d\vec{F} = \int -G \frac{mdM}{r^2} \hat{r} \]
Shell theorem

A uniform spherical shell of matter attracts a particle that is outside the shell as if all the shell’s mass were concentrated at its center.

\[ F = G \frac{mM}{r^2} \]

- \( m \) mass of particle
- \( M \) mass of shell
- \( r \) distance from particle to center of spherical shell

(http://hyperphysics.phy-astr.gsu.edu/hbase/mechanics/sphshell.html)
Consequence:

The gravitational force between two uniform spherical distributions of mass is the same as if all the mass of each sphere were concentrated at its center.
Gravitation near Earth’s surface

- Assume the Earth is exactly spherical.
- Assume uniform distribution of mass throughout the Earth.
- Neglect rotation effects.

Gravitational force on a particle near Earth’s surface:

\[ \vec{F}_g = -G \frac{mM}{r^2} \hat{r} \Rightarrow a_g = \frac{GM}{r^2} \]

\( r \) = distance between particle and center of the Earth.
Note: the free fall acceleration decreases with $r$

\[
a_g = \frac{GM}{r^2} = \frac{GM}{(R + h)^2}
\]

$h = \text{altitude} = \text{distance from particle to the surface of the Earth}$

<table>
<thead>
<tr>
<th>Altitude (km)</th>
<th>$a_g$ (m/s$^2$)</th>
<th>Altitude Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>9.83</td>
<td>Mean Earth surface</td>
</tr>
<tr>
<td>8.8</td>
<td>9.80</td>
<td>Mt. Everest</td>
</tr>
<tr>
<td>36.6</td>
<td>9.71</td>
<td>Highest crewed balloon</td>
</tr>
<tr>
<td>400</td>
<td>8.70</td>
<td>Space shuttle orbit</td>
</tr>
<tr>
<td>35700</td>
<td>0.225</td>
<td>Communications satellite</td>
</tr>
</tbody>
</table>
Rotation effects

Two forces act on this crate.

The normal force is upward.

The net force is toward the center. So, the crate's acceleration is too.

The gravitational force is downward.

\[ m\vec{a}_c = m\vec{a}_g + \vec{F}_N \]

\[ F_N = ma_g - m\omega^2 R \]

\[ g = a_g - \omega^2 R \]
i-Clicker

The gravitational acceleration on the surface of the Earth is $g$ (neglecting rotation.) What will it be on the surface of a planet that has half the mass of the Earth and half its radius?

A) $g/4$
B) $g/2$
C) $g$
D) $2g$
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A) $g/4$

B) $g/2$

C) $g$

D) $2g$

\[ g = G\frac{M}{R^2} \Rightarrow g' / g = M' R^2 / M (R')^2 = 4/2 = 2 \]
• Gravitation inside the Earth

A uniform shell of matter exerts no net gravitational force on a particle located inside it.

\[
\frac{r_1}{d_1} = \frac{r_2}{d_2} \quad \frac{A_1}{A_2} = \frac{r_1^2}{r_2^2}
\]

\[
\frac{F_1}{F_2} = \frac{A_1/d_1^2}{A_2/d_2^2} = 1
\]

\[
\vec{F}_1 + \vec{F}_2 = 0
\]
Suppose a very narrow tunnel is dug through the Earth along the North-South axis.

What is the gravitational force on a particle of mass $m$ at distance $r < R$ from the center?

Any thin spherical shell of matter of radius $r_{\text{shell}} > r$ does not yield any net force.

Only thin shells of radius $r_{\text{shell}} < r$ give a nonzero force.
The gravitational force on the particle is the same as the force due to a sphere of radius $r$.

Assume spherical shape and uniform mass distribution. Mass density

$$\rho = \frac{M}{4\pi R^3/3} = \frac{3M}{4\pi R^3}$$

$$F = G\frac{mM}{r^2} \times \text{Mass inside sphere of radius } r
\begin{align*}
&= \frac{Gm}{r^2} \times \left( \frac{4\pi r^3 \rho}{3} \right) = \frac{4\pi Gm\rho}{3}r = \frac{GmM}{R^3}r
\end{align*}$$
• All spherical shells are uniform and have the same mass $M$.
• Rank the situations according to the magnitude of the gravitational force on the particle.

A) $F_a > F_b > F_c$
B) $F_a = F_b > F_c$
C) $F_a < F_b < F_c$
D) $F_a < F_b = F_c$
i-Clicker

- All spherical shells are uniform and have the same mass $M$.

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$C) F_a < F_b < F_c$
$D) F_a < F_b = F_c$

$F_a = 0, F_b = F_c = GmM/r^2$
Gravitational potential energy

Work done by gravitational force: suppose a baseball moves upward from a distance \( r_1 \) to a distance \( r_2 > r_1 \) in the gravitational field of the Earth.

What is the work done by gravitational force?
Work is done as the baseball moves upward.

\[
W_{F_g} = \int_{r_1}^{r_2} \vec{F}_g \cdot d\vec{r}
\]

\[
= -GmM \int_{r_1}^{r_2} \frac{dr}{r^2}
\]

\[
= GmM \left( \frac{1}{r_2} - \frac{1}{r_1} \right)
\]

Does it depend on the path?
$W_{Fg}$ does not depend on the path.

Contribution of \textit{circular} arcs is 0:

$$\vec{F}_g \cdot d\vec{r} = 0$$

Contribution of \textit{radial} arcs:

$$\vec{F}_g \cdot d\vec{r} = -\frac{GmM}{r^2}dr$$

$W_{Fg,\text{curved path}} = W_{Fg,\text{radial path}}$
The gravitational force is conservative

Gravitational potential energy:

\[ U(\infty) - U(r) = -W_{Fg,r \to \infty} = \frac{GmM}{r} \]

\[ U(r) = -\frac{GmM}{r} \]
Gravitational potential energy: system of particles

- For any pair of particles $(i, j)$

\[ U_{ij} = -\frac{G m_i m_j}{r_{ij}} \]

\[ U_{\text{total}} = -\sum_{i<j} \frac{G m_i m_j}{r_{ij}} \]

\[ U_{\text{total}} = -\left(\frac{G m_1 m_2}{r_{12}} + \frac{G m_2 m_3}{r_{23}} + \frac{G m_1 m_3}{r_{13}}\right) \]
• **Escape speed**

How fast should a projectile be launched such that it escapes Earth’s gravitational field?

How fast should it be launched such that it does not fall back on Earth?
Find the initial speed of the rocket such that it escapes the Earth’s gravitational field, moving infinitely far away.

\[ A) \sqrt{\frac{MG}{R}} \]
\[ B) \sqrt{\frac{2MG}{R}} \]
\[ C) \sqrt{\frac{MG}{2R}} \]
\[ D) \text{It is impossible for the rocket to escape.} \]
i-Clicker

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A) $\sqrt{MG/R}$

B) $\sqrt{2MG/R}$

C) $\sqrt{MG/2R}$

D) It is impossible for the rocket to escape.
• Energy conservation

Initially: object on the surface of the Earth; kinetic and potential energy

\[ E_{mec} = K_0 + U_{grav} \]

Finally: object at \( \infty \); kinetic energy

\[ E_{mec} = K \geq 0 \]

\[ \frac{1}{2} m v_0^2 - \frac{G m M}{R} = \frac{1}{2} m v^2 \geq 0 \Rightarrow v_0 \geq \sqrt{\frac{2 G M}{R}} \]
\[ v_{\text{esc}} = \sqrt{\frac{2GM}{R}} \]

**Escape speed**
A projectile is launched horizontally from altitude \( h \) above the Earth’s surface. How fast should a projectile be launched such that it moves on a circular trajectory around the Earth?

\[ A) \sqrt{\frac{MG}{(R + h)}} \]
\[ B) \sqrt{\frac{2MG}{(R + h)}} \]
\[ C) \sqrt{\frac{MG}{2(R + h)}} \]

\[ D) \text{It is impossible for the projectile to circle the Earth.} \]
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C) \( \sqrt{\frac{MG}{2(R + h)}} \)

D) It is impossible for the projectile to circle the Earth.
\[ m\vec{a}_c = \vec{F}_g \]

\[ \frac{mv^2}{R + h} = \frac{GmM}{(R + h)^2} \]

\[ v = \sqrt{\frac{GM}{R + h}} \]

Orbital velocity