9. Center of Mass. Linear Momentum II

- Linear momentum for a system of particles

\[ \vec{P} = \sum_i \vec{p}_i = \sum_i m_i \vec{v}_i \]

\[ \vec{P} = M \vec{v}_{\text{com}}, \quad M = \sum_i m_i \]

\[ \frac{d\vec{P}}{dt} = \sum_i m_i \frac{d\vec{v}_i}{dt} = \sum_i m_i \vec{a}_i = \sum_i \vec{F}_i \]

\[ \frac{d\vec{P}}{dt} = \vec{F}_{\text{ext}} \]
Collision and Impulse

The figure depicts the collision at one instant. The ball experiences a force $F(t)$ that varies during the collision and changes the linear momentum of the ball.
The change in linear momentum of the ball is related to the force by Newton's second law:

\[ \vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \Delta \vec{p} = \int_{t_i}^{t_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F} dt \]

- **Impulse:**

\[ \vec{J} = \int_{t_i}^{t_f} \vec{F} dt \]
• The magnitude of $\vec{J}$ equals the area under the curve $F(t)$.

• Average force

$$\vec{F}_{\text{average}} = \frac{\vec{J}}{\Delta t}$$

• Newton’s 3rd law:

$$\vec{F}_{\text{ball}}(t) + \vec{F}_{\text{bat}}(t) = 0$$

at all times. Hence:

$$\vec{J}_{\text{bat}} = -\vec{J}_{\text{ball}}$$

$$|\vec{J}_{\text{bat}}| = |\vec{J}_{\text{ball}}|$$
A particle constrained to move on a horizontal smooth surface is subject to five forces as shown below. Which components of its momentum are conserved?

- **A)** $p_x, p_y$ are conserved
- **B)** $p_x$ conserved, $p_y$ not conserved
- **C)** $p_x$ not conserved, $p_y$ conserved
- **D)** $p_x, p_y$ not conserved.
Answer

A particle constrained to move on a horizontal smooth surface is subject to five forces as shown below. Which components of its momentum are conserved?

\[ F_x = \frac{dp_x}{dt} = 0 \]
\[ F_y = \frac{dp_y}{dt} = 8N \]

A) \( p_x, p_y \) are conserved

B) \( p_x \) conserved, \( p_y \) not conserved

C) \( p_x \) not conserved, \( p_y \) conserved

D) \( p_x, p_y \) not conserved.
i-Clicker

The graphs below encode the time dependence of the force magnitude for a body involved in a collision. How are the impulse magnitudes ordered?

A) \( J_a > J_b > J_c \)  \hspace{1cm} B) \( J_a < J_b < J_c \)

C) \( J_a = J_b > J_c \)  \hspace{1cm} D) \( J_a = J_b = J_c \)
i-Clicker

The graphs below encode the time dependence of the force magnitude for a body involved in a collision. How are the impulse magnitudes ordered?

\[ F \]
\[ 2F_0 \]
\[ 6t_0 \]
\[ t \]
\[ F \]
\[ 4F_0 \]
\[ 3t_0 \]
\[ t \]
\[ F \]
\[ 2F_0 \]
\[ 12t_0 \]
\[ t \]

(a) (b) (c)

\[ A) \ J_a > J_b > J_c \]
\[ B) \ J_a < J_b < J_c \]
\[ C) \ J_a > J_c < J_b \]
\[ D) \ J_a = J_b = J_c \]
• **Conservation of linear momentum**

  A system is **closed** if no particles leave or enter the system.

  A system is **isolated** if no external forces act on the system.

  **Isolated closed system:**

  \[
  \frac{d\vec{P}}{dt} = \vec{F}_{\text{ext}} = 0 \implies \vec{P} \text{ conserved}
  \]

  If no net external force acts on a closed system of particles, the total linear momentum \( P \) of the system cannot change.
Closed system:

\[(F_{\text{ext}})_x = 0 \Rightarrow \frac{dP_x}{dt} = 0 \Rightarrow P_x \text{ conserved}\]

\[(F_{\text{ext}})_y = 0 \Rightarrow \frac{dP_y}{dt} = 0 \Rightarrow P_y \text{ conserved}\]

\[(F_{\text{ext}})_z = 0 \Rightarrow \frac{dP_z}{dt} = 0 \Rightarrow P_z \text{ conserved}\]

If the component of the net external force on a closed system is zero along an axis, then the component of the linear momentum of the system along that axis cannot change.
**Example:** one dimensional explosion

A space hauler coupled to a space module moves with velocity \( \vec{v}_i = v_i \hat{i} \) relative to the Sun. The total mass of the system is \( M \) and the mass of the module is \( m < M \).

The module is ejected by a small explosion such that the relative velocity of the hauler with respect to the module is \( \vec{v}_{\text{rel}} = v_{\text{rel}} \hat{i} \).

Find the velocity of the hauler \( \vec{v}_{HS} \) relative to the Sun after the explosion.
Isolated closed system: $\vec{P}$ conserved.

$$\vec{P}_i = \vec{P}_f$$

$$P_{ix} = P_{fx}$$

$$P_{ix} = M v_{ix}$$

$$P_{fx} = (M - m) v_{HS,x} + m v_{MS,x}$$

$$\vec{v}_{HS} = \vec{v}_{MS} + \vec{v}_{rel}$$

$$v_{HS,x} = v_{MS,x} + v_{rel,x}$$
\[(M - m)v_{HS,x} + mv_{MS,x} = Mv_{ix}\]

\[v_{MS,x} = v_{HS,x} - v_{rel,x}\]

\[(M - m)v_{HS,x} + m(v_{HS,x} - v_{rel,x}) = Mv_{ix}\]

\[v_{HS} = v_{ix} + \frac{m}{M}v_{rel,x}\]
• **Example:** two dimensional explosion

A firecracker placed inside a coconut of mass $M$, initially at rest on a frictionless floor, blows the coconut into three pieces that slide across the floor.

Piece $C$, with mass $M_C = 0.3M$, has final speed $v_{fC} = 5.0 \text{ m/s}$.

(a) What is the speed of piece $B$, which has mass $M_B = 0.20M$?

(b) What is the speed of $A$?
Isolated closed system:

\[ \vec{P}_i = \vec{P}_f \]

\[ 0 = M_A \vec{v}_{fA} + M_B \vec{v}_{fB} + M_C \vec{v}_{fC} \]

\[ 0 = M_A v_{fA,x} + M_B v_{fB,x} + M_C v_{fC,x} \]

\[ 0 = M_A v_{fA,y} + M_B v_{fB,y} + M_C v_{fC,y} \]

x-axis: \[ M_A v_{fA} = M_B v_{fB} \cos \theta_B + M_C v_{fC} \cos \theta_C \]

y-axis: \[ 0 = M_B v_{fB} \sin \theta_B - M_C v_{fC} \sin \theta_C \]
\[ M_A v_f A = M_B v_f B \cos \theta_B + M_C v_f C \cos \theta_C \]

\[ M_B v_f B \sin \theta_B = M_C v_f C \sin \theta_C \]

\[ v_f B = \frac{M_C v_f C \sin \theta_C}{M_B \sin \theta_B} \]

\[ v_f A = \frac{M_B}{M_A} v_f B \cos \theta_B + \frac{M_C}{M_A} v_f C \cos \theta_C \]
i-Clicker

A golf ball of mass \( m \) moving with speed \( v_0 \) hits a bowling ball initially at rest. The golf ball bounces back with speed \( 0.9v_0 \). Let \( p_B \) denote the magnitude of the momentum of the bowling ball after collision. Which of the following statements is true?

\[
\begin{align*}
A) & \quad p_B = mv_0 \\
B) & \quad p_B = 0.9mv_0 \\
C) & \quad p_B = 0.1mv_0 \\
D) & \quad p_B = 1.9mv_0 
\end{align*}
\]
Answer

A golf ball of mass $m$ moving with speed $v_0$ hits a bowling ball initially at rest. The golf ball bounces back with speed $0.9v_0$. Let $p_B$ denote the magnitude of the momentum of the bowling ball after collision. Which of the following statements is true?

\[ p_B = \text{?} \]

A) $p_B = mv_0$

B) $p_B = 0.9mv_0$

C) $p_B = 0.1mv_0$

D) $p_B = 1.9mv_0$
\[ \vec{P}_i = \vec{P}_f \implies mv_0 = -0.9mv_0 + p_{B,x} \]

\[ p_{B,x} = 1.9mv_0 \]
Momentum and kinetic energy in collisions

Collisions in closed isolated system

- kinetic energy conserved $\Rightarrow$ elastic collisions

- kinetic energy not conserved, transferred to other forms of energy such as thermal energy $\Rightarrow$ inelastic collisions
• **Completely inelastic collisions in 1D**

**Completely inelastic:** the objects stick together after collision.
Here is the generic setup for an inelastic collision.

\[ \vec{P}_i = \vec{P}_f \]

\[ m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = m_1 \vec{v}_{1f} + m_2 \vec{v}_{2f} \]
In a completely inelastic collision, the bodies stick together.

\[ \vec{P}_i = \vec{P}_f \]

\[ m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = (m_1 + m_2) \vec{V} \]

\[ \vec{V} = \frac{m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i}}{m_1 + m_2} \]

\[ V_x = \frac{m_1 v_{1i,x} + m_2 v_{2i,x}}{m_1 + m_2} \]
The com of the two bodies is between them and moves at a constant velocity.

Here is the incoming projectile.

Here is the stationary target.

Collision!

The com moves at the same velocity even after the bodies stick together.

\[ \vec{V} = \vec{v}_{\text{com}} \]
An object at rest explodes into two pieces of unequal mass. One piece flies west at a speed $v$ and the second flies east at a speed $3v$. What is the velocity of the center of mass?

- A) 0
- B) $2v$ west
- C) $2v$ east
- D) cannot be determined
An object at rest explodes into two pieces of unequal mass. One piece flies west at a speed $v$ and the second flies east at a speed $3v$. What is the velocity of the center of mass?

- **A)** 0
- **B)** $2v$ west
- **C)** $2v$ east
- **D)** cannot be determined

\[
\vec{P}_i = \vec{P}_f = M\vec{v}_{\text{com}} = 0
\]
• **Example: ballistic pendulum**

  - large block of wood of mass $M = 5.4 \text{ kg}$ suspended from two long cords.
  - bullet of mass $m = 9.5 \text{ g}$ fired into the block.
  - system block + bullet swings upward a vertical distance $h = 6.3 \text{ cm}$.

  - $v_{\text{bullet}}$ ?
Step 1: • completely inelastic collision
• linear momentum conserved
• kinetic energy not conserved

\[ mv = (m + M)V \]
Step 2:

- upward swing
- linear momentum not conserved
- mechanical energy conserved

\[ E_{\text{mec}} = K + U = \text{constant} \]

\[ \frac{1}{2}(M+m)V^2 = (M+m)gh \]

\[ V = \sqrt{2gh} \]

\[ v = \frac{m + M}{m} \sqrt{2gh} \]
• **Example:** generic inelastic collision

A bullet of mass \( m = 10 \text{ g} \) moving directly upward at \( v = 1000 \text{ m/s} \) strikes and passes through the center of mass of a \( M = 5.0 \text{ kg} \) block initially at rest.

The bullet emerges from the block moving directly upward at \( v_1 = 400 \text{ m/s} \).

To what maximum height does the block then rise above its initial position?
**Step 1:**
- inelastic collision
- linear momentum conserved
- kinetic energy not conserved

\[ \vec{P}_i = \vec{P}_f \]

\[ mv = mv_1 + Mv_2 \]

\[ v_2 = \frac{m}{M}(v - v_1) \]
Step 2:  
- upward motion  
- linear momentum not conserved  
- mechanical energy conserved  
- assume the block does not move much during the collision

\[
\frac{1}{2} M v_2^2 = M g h
\]

\[
h = \frac{v_2^2}{2g} = \frac{m^2}{2M^2g}(v - v_1)^2
\]
• **Elastic collisions in 1D: both** linear momentum and kinetic energy are conserved.
• **Generic setup – stationary target**

![Generic setup diagram]

Here is the generic setup for an elastic collision with a stationary target.

**Before**

\[ m_1 \quad \vec{v}_{1i} \quad \vec{v}_{2i} = 0 \quad x \]

**Projectile**  \quad **Target**

**After**

\[ m_1 \quad \vec{v}_{1f} \quad \vec{v}_{2f} \quad x \]

\[ m_2 \]

- The linear momentum of the systems is conserved:
  \[ \vec{P}_i = \vec{P}_f \]

- The total kinetic energy of the system is conserved:
  \[ K_i = K_f \]

**Note:** the kinetic energy of each colliding body may change, but the total kinetic energy of the system does not change.
Here is the generic setup for an elastic collision with a stationary target.

Before

\[ m_1 \quad v_{1i} \quad m_2 \quad v_{2i} = 0 \quad x \]

Projectile \hspace{1cm} Target

After

\[ m_1 \quad v_{1f} \quad v_{2f} \quad x \]

\[
m_1 v_{1i} = m_1 v_{1f} + m_2 v_{2f}
\]

\[
\frac{1}{2} m_1 v_{1i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2
\]

\[
m_1 (v_{1i} - v_{1f}) = m_2 v_{2f}
\]

\[
m_1 (v_{1i} - v_{1f})(v_{1i} + v_{1f}) = m_2 v_{2f}^2
\]

\[
\downarrow
\]

\[
v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}
\]

\[
v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}
\]