We learned last time that the behavior of light, of electromagnetic radiation in general, can not be understood unless we acknowledge the energy comes in lumps, lumps we call "quanta", or "photons", with energy and momentum:

\[ E = hf \quad p = \frac{E}{c} = \frac{hf}{c} = \frac{h}{\lambda} \]  \hspace{1cm} (1)

The appearance of light as little lumps of energy has profound and disturbing consequences, for we need to reconcile it with the fact that light also produces interference patterns.
Today we will look at the consequences of quantum mechanics for electrons. Remarkably, electrons also exhibit interference, a fact first guessed by the French physicist Louis de Broglie, and later confirmed at Bell Labs, in New York City, by Davisson & Germer.

These experiments tell us that like light, electrons have a wavelength

\[ \lambda = \frac{\hbar}{p} \]  

But how do they arrive as "lumps" or particles, yet also interfere? The answer turns out to be profound: we are forced to regard light & electrons as probability waves.
In 1924 Louis de Broglie proposed that electrons are matter waves, with a wavelength

\[ \lambda = \frac{h}{p} \]  

His reasons for making this proposal were connected with the energy levels of the hydrogen atom, something we will come back to later. One of the consequences of this result is that electrons diffract—indeed, they scatter off crystals just like X-ray photons, obeying the Bragg law

\[ 2d \sin \Theta = n\lambda \]

where \( \lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} \), where \( E \) is the electron energy. Indeed, under the right circumstances, electrons, like light can be made to exhibit a double slit interference pattern.
**Feynman's Discussion**  
(Feynman Lectures on Physics I: 37 or III: 1)

**Bullets**
- Arrive one at a time
  - $N_{12}(x) = N_1(x) + N_2(x)$
- No interference

**Water Waves**
- Arrive continuously
  - $h = h_1 + h_2$
  - $I_{12} = (h_1 + h_2)^2 
eq I_1 + I_2$
  - Interference

$$I = 2I \cos^2 \left( \frac{\phi}{2} \right)$$
Electrons (or photons)

- Arrive one at a time, like **bullets**
- Yet \[ N_{12}(x) \neq N_1(x) + N_2(x) \]

If you **close** one slit, or if you "**look**" which slit the electron goes through, the interference disappears.

- \[ N_{12}(x) = |\psi_1(x) + \psi_2(x)|^2 \]

\( \psi \) is a probability wave: a "**wavefunction**" whose square determines the probability of finding an electron (or other particle).
e.g. Calculate the de Broglie wavelength of

(a) A 1 keV electron

(b) A 1 keV photon

(c) A 100 kg sprinter moving at 10 m/s.

(a) An electron of momentum $p$ has energy $E = \frac{p^2}{2m} \Rightarrow p = \sqrt{2mE}$

Thus $\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} = \frac{h}{\sqrt{2m \text{ eV}}}$

$$= \frac{6.63 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 1.602 \times 10^{-19} \times 10000}}$$

$$= 3.88 \times 10^{-11} \text{ m}$$

(b) A photon has $\lambda = \frac{h}{p} = \frac{hc}{E} = \frac{4.13 \times 10^{-15} \text{ eVs} \times 3 \times 10^8}{1000 \text{ eV}}$

$$p = \frac{E}{c} = 1.24 \times 10^{-9} \text{ m}$$

(c) A runner with momentum $p = 100 \text{ kg} \times 10 \text{ m/s} = 10^3 \text{ kg \cdot m/s}$

has de Broglie $\lambda = \frac{6.6 \times 10^{-34}}{10^3} = 6.6 \times 10^{-37} \text{ m}$!
Waves on a string, electromagnetic waves in space, sound waves, all satisfy a "wave" equation. E.g waves on a string

$$h(x,t)$$

$$\frac{\partial^2 h}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 h}{\partial t^2} = 0$$

E.g. $$h(x,t) = h_0 \cos(kx - \omega t)$$

$$\frac{\partial^2 h}{\partial x^2} = -k^2 h$$

$$\Rightarrow h(-k^2 + \omega^2/c^2) = 0$$

$$\frac{\partial^2 h}{\partial t^2} = -\omega^2 h$$

$$\Rightarrow k = \frac{\omega}{c}$$

$$\frac{2\pi}{\lambda} = \frac{2\pi f}{c}$$

But what about electron waves: what is their wave equation?

This question was asked by Peter Debye, talking to Erwin Schrödinger in the winter of 1926.

"Schrödinger, you are not working on anything important right now. Why don't you tell us some time about the ideas of de Broglie, which seems to have attracted some attention."
Schrödinger returned some weeks later, with his famous equation.

The basic idea of his equation is a complex wavefunction \( \Psi(x,y,z,t) \), which is a complex number that can be written in the form \( a + ib \), where \( i^2 = -1 \) and \( a \) & \( b \) are real numbers. In most cases the space & time variables can be separated, and \( \Psi \) factorizes into the form

\[
\Psi(x,y,z,t) = \Psi(x,y,z) e^{-i\omega t}
\]

(related to energy by \( E = hf = (h/2\pi)\omega \).)

The probability of detecting a particle in a small volume centered on a given point in a matter wave is proportional to \( |\Psi(x)|^2 \) at that point.
In its simplest form, Schrödinger's equation in one dimension takes the form

$$\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2} (E - U(x))\psi = 0$$

(4)

where $E$ is the mechanical energy of the particle, $U(x)$ is the potential energy and $\hbar = h/2\pi = 1.054\times10^{-34}$ Js is often called "hbar". We recognize $E - U(x) = K$ the kinetic energy. Classically

$$E - U = \frac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{p^2}{2m} = \frac{1}{2m}\left(\frac{\hbar}{\lambda}\right)^2$$

So in terms of the wavenumber $k = 2\pi/\lambda$

$$(E - U) = \frac{1}{2m}\left(\frac{\hbar k}{2\pi}\right)^2 = \frac{\hbar^2k^2}{2m}$$

(5)

Substituting into (4) we obtain

$$\frac{d^2\psi}{dx^2} + k^2\psi = 0$$

Schrödinger's equation

\[\text{Constant } U(x) = U\]

(6)

where

$$k = \frac{\sqrt{2m(E - U)}}{\hbar}$$

(7)
The simplest solution is

$$\Psi(x) = e^{ikx}$$

You can check that

$$\frac{d^2\Psi}{dx^2} = (ik)^2 e^{ikx} = -k^2 \Psi.$$ A more general solution is

$$\Psi(x) = A e^{ikx} + B e^{-ikx}$$

**Probability density**  \( \Psi = Ae^{ikx} \)

To calculate the probability density for \( \Psi = Ae^{ikx} \), let's write

$$|\Psi|^2 = |Ae^{ikx}|^2 = A^2 e^{ikx} e^{-ikx}$$

But since \( e^{i\theta} = \cos \theta + i \sin \theta \), \( e^{ikx} = \cos kx + i \sin kx \), \( (e^{ikx})^* = e^{-ikx} \),

so

$$|e^{ikx}|^2 = (e^{ikx})^* e^{ikx} = e^{-ikx} e^{ikx} = e^0 = 1.$$

So the probability density of a single right-moving wave is

$$P = A^2 = \text{constant}$$

This is because the particle has a constant potential energy.
HEISENBERG'S UNCERTAINTY RELATION

In one dimension, the uncertainty principle is

\[ \Delta x \Delta p_x \geq \hbar \]

with a corresponding relation for the y & z axes.

Heisenberg's relation means that the more accurately one measures position, the more uncertain the momentum becomes. This is connected with diffraction of the matter wave. If I send a wave through a slit of width \( a = \Delta x \) I diffract the wave, spreading its diffraction pattern over an angular width \( \Delta \theta \sim \sin \Theta = \frac{\lambda}{a} = \frac{\lambda}{\Delta x} \). But the uncertainty in momentum is then \( \Delta p_x \sim p \sin \Theta = \left( \frac{\hbar}{\lambda} \right) \frac{\lambda}{\Delta x} \), so that \( \Delta x \Delta p_x \geq \hbar \). A more careful calculation gives \( \Delta x \Delta p_x \geq \hbar \).
\[ p = \hbar \psi \]
\[ \Delta p_x \sim p \sin \theta \]
\[ a \sin \theta = \frac{\lambda}{\Delta x} \]
\[ \Rightarrow \Delta p_x = p \left( \frac{\lambda}{\Delta x} \right) = \frac{\hbar}{\lambda} \left( \frac{\lambda}{\Delta x} \right) = \hbar \]
\[ \Delta x \Delta p_x \geq \hbar \quad \text{Better:} \quad \frac{\hbar}{2\Delta x} = \kappa \]

---

e.g. An electron has speed \( v = 2.05 \times 10^6 \text{ m/s} \pm 0.5\% \). What is the minimum uncertainty in position?

\[ p_x = m v_x = 9.11 \times 10^{-31} \text{ kg} \times (2 \times 10^6 \text{ m/s}) = 1.87 \times 10^{-24} \text{ kg m/s} \]
\[ \Delta p_x = 5 \times 10^{-3} \times p_x \approx 9.35 \times 10^{-24} \text{ kg m/s} \]
\[ \Delta x \geq \frac{\hbar}{\Delta p_x} = \frac{1.054 \times 10^{-34} \text{ Js}}{9.35 \times 10^{-24} \text{ kg m/s}} = 0.11 \times 10^{-7} \text{ m} = 1.1 \text{ nm} \]