Sofar, we've learnt that with Einstein's postulate, that the laws of physics, including electromagnetism, are the same for all inertial observers. This means we must adopt a new view of space and time, one which abandons the concept of universal simultaneity.

We saw that by observing a light clock in two different inertial reference frames, that the time interval in \( t_0 \) is the rest frame, and the time interval in the moving frame \( t \) were related by

\[
    t = t_0 \gamma, \quad \gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}
\]
\[ \Delta T = \frac{2l_y}{c} \]

\[ \Rightarrow \Delta T^2 = \Delta t^2 - \frac{v^2}{c^2} \Delta t^2 \]

We also see that if we use the light clock to mark notches along a moving ruler, that the length measured by the man in the frame of the clock was \( l = V \Delta T \), whereas the length measured between the notches in the frame of the ruler was \( \Delta t = l_0 \), so that

\[ l = V \Delta T = \frac{V \Delta t}{\gamma} = V \Delta t \sqrt{1 - \frac{v^2}{c^2}} = l_0 \sqrt{1 - \frac{v^2}{c^2}} \]

Length contraction
Now we're going to put his new knowledge to work to derive the transformation that links different reference frames: the "Lorentz Transformation". We are going to go on to examine the consequences for velocity transformations, for the Doppler shift of light frequencies and for kinematics itself.
LORENTZ TRANSFORMATION

"Moving" Frame

\[ x' = \text{length of measuring rod} = l_0 \]

LAB FRAME

\[ x = vt + l \Rightarrow l = x - vt \]

\[ \text{moving rod of length } \quad l = l_0 \sqrt{1 - \frac{v^2}{c^2}} \]

But

\[ l = l_0 \sqrt{1 - \frac{v^2}{c^2}} = x' \sqrt{1 - \frac{v^2}{c^2}} \]

\[ \Rightarrow \quad x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}} = \gamma (x - vt) \]
But, viewed from the moving frame, the lab frame is moving backwards at velocity \(-v\). Treating the moving frame as the lab frame, by relativity we must also be able to write

\[
x = \frac{(x' + vt')}{\sqrt{1 - \frac{v^2}{c^2}}} \quad \text{(Reverse Lorentz Transformation)}
\]

But \(x' = \delta(x - vt)\), so

\[
x = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left( x - vt \right) + \frac{vt'}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

Reorganizing \(x\) & \(t\)

\[
x \left( 1 - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right) + \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} \quad t = \frac{vt'}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

\[
x \left( \frac{-v^2}{\sqrt{1 - \frac{v^2}{c^2}}} \right) + \frac{vt}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{vt'}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

\[
\Rightarrow \quad t' = t - \frac{vx/c^2}{\sqrt{1 - \frac{v^2}{c^2}}}
\]
Written out in short-hand

\[
\begin{align*}
x' &= \gamma(x-\nu t) \\
t' &= \gamma(t-\nu x/c^2) \\
y' &= y \\
z' &= z
\end{align*}
\]

\[
\gamma &= \frac{1}{\sqrt{1 - \nu^2/c^2}}
\]

Lorentz Transformation.

- If we use units where \( c = 1 \) (e.g. feet & nanoseconds)

\[
\begin{align*}
x' &= \gamma(x-\nu t) \\
t' &= \gamma(t-\nu x)
\end{align*}
\]

\( \gamma \) Symmetric

- You can verify that

\[
x'^2 - c^2t'^2 = x^2 - c^2t^2
\]

or more generally

\[
x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2
\]

- Set \( x=0 \), \( t'=\gamma t \) \underline{Time dilation}

- Set \( x'=l_0 \), \( t=0 \) \( \Rightarrow \frac{l_0}{\gamma} = l = l_0\sqrt{1 - \nu^2/c^2} \) \underline{length contraction}
**37.5b Relativistic Velocity Transformation**

If we differentiate the Lorentz transformation, then

\[
\frac{dx'}{dt'} = \frac{\frac{dy}{dt} - u}{1 - \frac{u}{c^2} \frac{dx}{dt}}
\]

\[
dx' = Y (dx - v dt)
dt' = Y (dt - \frac{v}{c^2} dx)
\]

- If \( v \) & \( u \) are much smaller than \( c \), \( v, u << c \), then \( u' = u - v \)
- If \( u = c \), then
  \[
  u' = \frac{c - v}{1 - \frac{uv}{c^2}} = \frac{c - v}{1 - \frac{v}{c}} = c
  \]
- If we change \( u \rightarrow u', v \rightarrow v' \) then
  or solve for \( u \)
  \[
  u = \frac{u' + v}{1 + \frac{uv}{c^2}}
  \]
37.5b Example

\[ u = \frac{u' + v}{\frac{1}{1 + \frac{uv}{c^2}}} = \frac{0.7c + 0.9c}{1 + \frac{(0.7c)(0.9c)}{c^2}} = 0.982c. \]
37.6 Doppler Effect for E.M Waves

When a source of sound moves towards us, it acquires a higher pitch. When it moves away from us, it has a lower pitch. This is called the Doppler effect. The same effect also occurs for electromagnetic waves, but now we must take into account the effect of time dilation.

\[ \lambda = \frac{c}{v} T \]

\[ f = \frac{c}{\lambda} = \left( \frac{c}{c-v} \right) \frac{1}{T} \]  \hspace{1cm} (1)

But \[ T = \frac{T_0}{\sqrt{1-\frac{v^2}{c^2}}} \]

\[ \frac{1}{T} = \frac{1}{T_0} \sqrt{1-\frac{v^2}{c^2}} = f_0 \sqrt{1-\frac{v^2}{c^2}} \]  \hspace{1cm} (2)
Combining (1) & 2

\[ f = \frac{c}{c-v} \sqrt{\frac{c^2-v^2}{c^2}} f_0 \]

\[ = \frac{\sqrt{c^2-v^2}}{\sqrt{(c-v)^2}} f_0 \]

\[ = \frac{\sqrt{(c-v)(c+v)}}{\sqrt{(c-v)^2}} f_0 \]

\[ f = \frac{\sqrt{c+v}}{\sqrt{c-v}} f_0 \]  

E.M. 
DOPPLER

Moving Towards You

Moving Away From You
e.g. Blue light of wavelength $\lambda = 400\text{nm}$ is emitted from a moving source. An observer measures its wavelength to be $\lambda' = 800\text{nm}$. How fast is the source moving relative to the observer?

\[ f_0 = \frac{c}{\lambda_0} = \frac{3 \times 10^8}{4 \times 10^2} = 0.75 \times 10^{15} = 750 \text{THz} \]

\[ f' = \frac{1}{2} f_0 = \sqrt{\frac{c+v}{c-v}} f_0 \]

\[ \frac{1}{2} = \sqrt{\frac{c+v}{c-v}} \quad \frac{1}{4} = \frac{c+v}{c-v}. \]

\[ \Rightarrow (c-v) = 4(c+v) \]

\[ -3c = 5v \quad \Rightarrow \quad v = -0.6c \]

Source is moving away from observer at $0.6c$.  

37.7 Relativistic Momentum

In a collision, the total mass & the total momentum are conserved. If this is to remain true for all inertial observers, we need to reuse what we mean by momentum. It turns out that the form that guarantees momentum conservation holds for all inertial observers is

\[ \mathbf{p} = \frac{m_0 \mathbf{v}}{\sqrt{1 - v^2/c^2}} = \gamma m_0 \mathbf{v} \]

where \( m_0 \) is the (rest) mass of the particle.
Newton's law of motion now becomes

\[ F = \frac{d}{dt} \left( \frac{m_0 \vec{V}}{\sqrt{1 - \frac{V^2}{c^2}}} \right) \]

If acceleration is in the same direction as motion

\[ F = \frac{m_0 a}{\sqrt{1 - \frac{V^2}{c^2}}} + \frac{m_0 V}{(1 - \frac{V^2}{c^2})^{3/2}} \frac{V a}{c^2} \]

\[ = \frac{m_0 a}{(1 - \frac{V^2}{c^2})^{3/2}} = \gamma^3 m_0 a \]

\[ a = \frac{F}{m_0} \left( 1 - \frac{V^2}{c^2} \right)^{3/2} \]

\[ m_{rel} = \frac{m_0}{\sqrt{1 - \frac{V^2}{c^2}}} \]

\[ p = \gamma m_0 v \]

\[ F = \gamma^3 m_0 a \]
But note $\mathbf{F} \neq \text{mrel} \mathbf{a}$, $K.E \neq \frac{1}{2} \text{mrel} v^2$

Motion in a circle, $\gamma = \text{constant}$. \[ \mathbf{F} \perp \mathbf{v} \]

\[ \mathbf{F} = \frac{m_0}{\sqrt{1-v^2/c^2}} \frac{d\mathbf{v}}{dt} = \text{mrel} \mathbf{a} \quad \text{only if} \quad \mathbf{F} \perp \mathbf{v} \]

In general $\mathbf{F}$ & $\mathbf{a}$ are not parallel.
Calculate momentum & acce. of e⁻ moving at 0.9c in an electric field of 5x10⁵ N/C

\[ F = qE = (1.6 \times 10^{-19} \text{C}) \times 5 \times 10^5 \text{N/C} = 8 \times 10^{-14} \text{N} \]

\[ \gamma = \frac{1}{\sqrt{1-(0.9)^2}} = 2.29 \]

\[ p = \gamma m_0 u = 2.29 \times (9.11 \times 10^{-31} \text{kg}) \times 0.9 \times 3 \times 10^9 \text{m/s} \]

\[ = 5.6 \times 10^{-22} \text{ kg m/s} \]

\[ a = \frac{F}{\gamma^3 m_0} = \frac{8 \times 10^{-14}}{(2.29)^3 (9.11 \times 10^{-31})} = 7.3 \times 10^{15} \text{ m/s}^2 \]
How much work do we do in accelerating a relativistic particle? Classically, the work done is the change in kinetic energy, where \( K.E = \frac{1}{2}mv^2 \). What is it relativistically?

\[
W = \int F \, dx = \int \frac{m_0 a}{(1 - v^2/c^2)^{3/2}} \, dx \quad (F \parallel v)
\]

Now \( a \, dx = \frac{d}{dt} \frac{dx}{dt} = \frac{du}{dt} \frac{dx}{dt} = dV \, v = \frac{1}{2} \, dV^2 \). So

\[
W = \frac{m_0}{2} \int_{v_i}^{v_f} \frac{dV^2}{(1 - v^2/c^2)^{3/2}}
\]

\( v^2/c^2 = X \Rightarrow dx = \frac{1}{c^2} \, dV^2 \)

\[
W = \frac{m_0c^2}{2} \int \frac{dx}{(1-X)^{3/2}} = m_0c^2 \left[ \frac{1}{\sqrt{1-v_f^2/c^2}} - \frac{1}{\sqrt{1-v_i^2/c^2}} \right]
\]

\[= E_f - E_i \]
where

\[
E(u) = m_0 c^2 \left( \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right)
\]

We may interpret \( E_0 = m_0 c^2 \) as the rest energy. For small \( u \)

\[
E(u) = m_0 c^2 \left[ 1 - \frac{1}{2} \left( \frac{v^2}{c^2} \right) + \frac{3}{8} \left( \frac{v^4}{c^4} \right) + \ldots \right]
\]

\[
E(u) = m_0 c^2 + \frac{1}{2} m_0 v^2 + \frac{3}{8} m_0 v^4 / c^2 + \ldots
\]

REST ENERGY  CLASSICAL K.E  FIRST RELATIVISTIC CORRECTION.
a) What is the rest energy of an electron in electron volts?

$$m_0c^2 = 9.109 \times 10^{-31} \times (2.998 \times 10^8)^2$$

$$= 8.187 \times 10^{-14} \text{ J}$$

$$m_0c^2 = eV \quad V = \frac{m_0c^2}{e} = \frac{8.187 \times 10^{-14}}{1.602 \times 10^{-19}}$$

$$= 5.11 \times 10^5 \text{ eV}$$

$$= 0.511 \text{ MeV}$$

b) How fast is an e-maring after accelerating through a potential of 5 MeV?

$$m_0c^2(\gamma - 1) = 5 \times 10^6 e$$

$$\gamma - 1 = 5 \times 10^6 \left( \frac{e}{m_0c^2} \right) = \frac{5}{0.511} = 9.78$$

$$\gamma = 10.78 \quad \frac{1}{\sqrt{1 - \frac{1}{\gamma^2}}} \quad \Rightarrow \quad \frac{1}{1 - v/c^2} \Rightarrow \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}}$$

$$\frac{v}{c} = \sqrt{1 - \frac{1}{10.78^2}} = 0.996 \quad \Rightarrow \quad v = 0.996c$$
**Velocities**

\[ u = \frac{dx}{dt} \]

\[ u' = \frac{dx'}{dt'} \]

Gallilean

\[ u' = u - v \]

\[ t' = t \]

\[ \frac{dx'}{dt'} = \frac{dx - vt}{dt} \]

\[ \frac{dx'}{dt'} = \frac{1}{\gamma} \frac{dx}{dt} \]

Relativistic

\[ dx' = (dx - vt) \gamma \]

\[ dt' = (dt - \frac{v}{c^2} dx) \gamma \]

\[ \frac{dx'}{dt'} = \frac{dx - vt}{dt - \frac{v}{c^2} dx} = \frac{\frac{dx}{dt} - v}{1 - \frac{v^2}{c^2} \frac{dx}{dt}} \]

\[ u' = \frac{u - v}{1 - \frac{uv}{c^2}} \]

**Momentum**

\[ \vec{p} = m_0 \frac{dx}{dt} \rightarrow m_0 \gamma \frac{dx}{dt} = m_0 \gamma \frac{dx}{d\tau} \]

\[ d\tau = dt \gamma \]

\[ F = m_0 \frac{d\gamma v}{dt} = m_0 \left( \gamma \frac{dv}{dt} - \frac{\gamma^2 v^2}{c^2} \frac{dv}{d\tau} \right) \]

\[ F = m_0 \gamma^2 \frac{dv}{d\tau} \]

\[ W = \int F \cdot dx = \int m_0 \gamma^3 \frac{d\gamma v}{d\tau} \cdot \frac{dx}{d\tau} d\tau = \int m_0 \gamma^3 \frac{dv}{d\tau} v d\tau \]

\[ W = m_0 \int \frac{\gamma^3 v dv}{\sqrt{\frac{1}{c^2} - \frac{v}{c^2}}} = m_0 c^2 \int \gamma' = m_0 c^2 (\gamma - 1) \]

\[ \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ \frac{dx'}{dx} = \frac{1}{\gamma} \left( \frac{2}{c^2} \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \right) = \frac{v}{c} \left( \sqrt{\frac{1}{c^2} - \frac{v}{c^2}} \right) \]

\[ W = m_0 c^2 = mc^2 \]
Various equations

\[ k = m_0(Y - 1)c^2 \]

\[ Y = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = 1 + \frac{v^2}{2c^2} + \frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{v^4}{c^4}\right) + \ldots \]

\[ = 1 + \frac{v^2}{c^2} + \frac{3}{8} \frac{v^4}{c^4} \]

\[ k = m_0c^2 \left(\frac{v^2}{2c^2} + \frac{3}{8} \frac{v^4}{c^4} + \ldots\right) = \frac{m_0v^2}{2} + \frac{3}{8} m_0 \frac{v^4}{c^2} + O\left(\frac{v^6}{c^4}\right) \]

\[ = \frac{m_0v^2}{2} \left(1 + \frac{3}{8} \frac{v^2}{c^2} \ldots\right) \]

\[ E = m_0 Yc^2 \]

\[ p = m_0 Yv \]

\[ E^2 = m_0^2 c^4 \frac{1}{1 - \frac{v^2}{c^2}} \]

\[ p^2 c^2 = m_0^2 c^4 \frac{v^2 c^2}{(1 - \frac{v^2}{c^2})} \]

\[ E^2 - p^2 c^2 = m_0^2 c^4 \]

\[ E = \sqrt{(pc)^2 + (m_0^2 c^4)} \]

\[ (k + m_0c^2)^2 = (pc)^2 \quad \Rightarrow \quad k^2 + 2k m_0c^2 + (m_0c^2)^2 = (pc)^2 \]

\[ (E - m_0c^2)^2 = (pc)^2 \quad = (E - m_0c^2)(E + m_0c^2) = k(k + 2m_0c^2) = (pc)^2 \]

- If \( m_0 = 0 \quad E = pc \quad \text{Photons, neutrons (almost), gravitons...} \]

- \( E = \sqrt{(pc)^2 + (m_0^2 c^4)} \)