Physics 228

Today:
Particles as Waves
Bohr Atom
Uncertainty Principle: Energy and Time

We have just seen that $p$ and $x$ are "conjugate" variables. But our equation for the traveling wave

$$E = E_0 \sin\left(\frac{px}{\hbar} - \frac{Et}{\hbar}\right)$$

has two pairs of arguments, $(p, x)$ and $(E, t)$. What about localization in energy and time?

Answer: Energy - time uncertainty applies:

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

This is relevant for excited states (or particles) with a finite lifetime $\Delta t$. The excited state energy (or the particle’s mass, via $E = mc^2$) is then subject to the above uncertainty.
In a two-slit interference experiment, a single photon is detected in the upper half of the screen. The photon must have passed through

a. the upper slit
b. the lower slit
c. neither slit
d. both slits
e. impossible to decide which slit
Quantum Weirdness

• Wave-Particle Duality: Waves are also particles. Particles are also waves.

• Uncertainty Principle: Precise knowledge of position and momentum of a photon, at the same time, is not possible. This applies to all particles, not just photons.

• Rather than telling us what will definitely happen in a given experiment, quantum mechanics only tells us the probability of certain outcomes.

• In the two-slit experiment, we cannot say which slit a photon goes through. If there is interference, it goes through both.

• We will see that this is true for electrons and other particles as well.
Wave Particle Duality

In 1924, Louis de Broglie proposed that, since light acts as both a particle and a wave, classical particles such as electrons also act as both particles and waves. “Matter Waves”

For matter waves, we have the same relations between momentum $p$ and wavelength $\lambda$ as for light:

$$ p = \frac{h}{\lambda} $$

The uncertainty relations we discussed for photons - $\Delta E \Delta t \geq \frac{\hbar}{2}$, $\Delta x \Delta p_x \geq \frac{\hbar}{2}$, etc. also apply to particles.

But we have different relations between energy $E$ and momentum $p$ since the photon is massless:

- photons: $E = pc$
- Massive particles (relativistic): $E^2 = (pc)^2 + (mc^2)^2$
- Massive particles (non-relativistic): $K = p^2/2m$
## Wave Particle Duality Summary

<table>
<thead>
<tr>
<th>Photons</th>
<th>Electrons, etc.</th>
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<tbody>
<tr>
<td>$E = hf$</td>
<td>$E = hf$</td>
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<tr>
<td>$p = h/\lambda = \hbar k$</td>
<td>$p = mv = h/\lambda = \hbar k$</td>
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<tr>
<td>$p = E/c$</td>
<td>$p = (2mK)^{1/2}$</td>
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Experimental Confirmation

The wave nature of electrons was demonstrated within a few years of its prediction by Davisson and Germer, by measuring diffraction of electrons by the atoms on the surface of a Ni crystal:

1. A heated filament emits electrons.
2. The electrons are accelerated by electrodes and directed at a crystal.
3. Electrons strike a nickel crystal.
4. The detector can be moved to detect scattered electrons at any angle $\theta$. 
Electron Diffraction

Scattering peaks are seen at diffraction angles given by

\[ m \lambda = d \sin \theta. \]

\( m \) is an integer (diffraction order)
\( \lambda \) is the wavelength given by \( \lambda = \frac{h}{p} \)
\( d \) is the separation between atom rows on the surface.

At low energies, the electrons scatter off the surface, and the surface atoms act as a diffraction grating.

At high energies, the electrons penetrate the bulk of the crystal, which acts as a 3D diffraction grating, just like for x-ray diffraction in Bragg scattering:

\[ m \lambda = 2d \sin \theta. \]
What is the wavelength of an electron, moving at \( v = 0.9 \, c \).

\[ p = \gamma m v = 5.64 \times 10^{-23} \, \text{kg} \cdot \text{m/s}. \]

\[ \lambda = \frac{h}{p} = 6.626 \times 10^{-34} \, \text{Js} / 5.64 \times 10^{-23} \, \text{kg} \cdot \text{m/s} \approx 10^{-11} \, \text{m}. \]

Remember the diffraction limit for imaging:

\[ \sin \theta = 1.22 \, \lambda / D. \]

Since the focal length of a lens cannot be made much smaller than the diameter, the smallest object that can be resolved in a microscope is about the size of the wavelength used.

Electron wavelengths are similar to atomic sizes, so electrons are useful for imaging atoms: Electron microscope.

The size of an atom is \( \approx 10^{-10} \, \text{m} \).

The size of an atomic nucleus is \( \approx 10^{-15} \, \text{m} \).

The size of an electron is zero (point particle, for all we know).
Electron Microscope

- Similar to optical microscope in principle, just replace light by electrons.
- For lenses, use appropriately shaped electric fields.
- Useful for imaging atoms, cells, viruses, nanostructures, ...

![Diagram of Electron Microscope](image)

- High-voltage supply
- Cathode
- Accelerating anode
- Condensing lens
- Objective lens
- Intermediate image
- Projection lens
- Final image
- Photographic film or fluorescent screen

![Image of Electron Microscope](image)

- Eu atom
- 2 nm
What is your wavelength, as you are running along at 10 m/s?

\[ p = mv = 70 \text{ kg} \times 10 \text{ m/s} = 700 \text{ kg}\cdot\text{m/s}. \]

\[ \lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34} \text{ Js}}{700 \text{ kg}\cdot\text{m/s}} \approx 10^{-36} \text{ m}. \]

Can we observe interference/diffraction?

a. Sure. Just give me a fine enough diffraction grating.

b. Two slit interference would require you to run through two doors at the same time. This is only possible in science-fiction movies.

c. This works only for people with split-personality disorder.

d. Wave-particle duality applies only to microscopic particles, not to macroscopic bodies.

e. Impossible to observe in practice. Wavelength is many orders of magnitude smaller than size of human body.
Towards the Structure of the Atom

In 1897, J. J. Thomson had discovered the electron and measured its charge/mass ratio.

By 1909, R Millikan measured the electron charge by sticking electrons to much larger oil drops, and observing their motion under the forces of gravity and an electric field.

The resulting electron mass was much smaller than the mass of atoms. Thus, almost all the mass of the neutral atom is associated with its positive charge component, not with the negatively charged electron.
In 1911, E. Rutherford led experiments scattering alpha particles ($^4\text{He}$ nuclei) from gold atoms. Most of the α particles went straight through the gold foil, while a very small number were deflected at large angles:
Rutherford concluded that the positive charge (and most of the mass) of the atom is concentrated in a very compact region, the atomic “nucleus”.
Planetary Model of Atom

- positively charged, massive nucleus
- negatively charged, light electrons
- electrostatic attraction
- Together, these facts suggest electrons “orbiting” around the nucleus, like planets around the sun.
- Problem: The orbiting electrons would radiate light, carrying away energy
- Electrons would spiral into nucleus, atom would collapse.

ACCORDING TO CLASSICAL PHYSICS:
- An orbiting electron is accelerating, so it should radiate electromagnetic waves.
- The waves would carry away energy, so the electron should lose energy and spiral inward.
- The electron’s angular speed would increase as its orbit shrank, so the frequency of the radiated waves should increase.
Thus, classical physics says that atoms should collapse within a fraction of a second and should emit light with a continuous spectrum as they do so.

IN FACT:
- Atoms are stable.
- They emit light only when excited, and only at specific frequencies (as a line spectrum).
Bohr Model of Hydrogen Atom

Since electrons behave as waves, Niels Bohr reasoned that in the atom they might form standing waves, or resonances, just like standing waves on a string:
Bohr proposed that only those electron orbits are allowed where an integer number of wavelengths fit into a circular orbit:
Bohr’s hypothesis explains the stability of atoms, as well as the line spectra:

Electrons absorb or emit particular energy photons to transition between two orbits, but once they are in the lowest energy orbit they cannot emit any more energy.

Consider: $2\pi r_n = n\lambda_n$ where $n$ is the "principal quantum number".

From de Broglie, $p = h/\lambda$, we obtain $2\pi r_n = n\lambda_n = nh/p_n$. This means that the angular momentum $L_n = r_n p_n = nh/2\pi = n\hbar$.

An integral number of wavelengths in each orbit leads to an orbital angular momentum that is an integer multiple of $\hbar$. Angular momentum is "quantized".
Bohr Atom Radius

- Consider an electron in circular orbit about a proton of radius \( r_n \). The circumference is \( 2\pi r_n \).

- The electrostatic attraction force between the electron and proton is \( F = e^2 / 4\pi \varepsilon_0 r_n^2 \).

- In a circular orbit, this force must be equal to the mass times the centripetal acceleration \( m v_n^2 / r_n \).

- Multiply both sides of the force equation by \( m r_n^3 \) to obtain:
  \[
  m e^2 r_n / 4\pi \varepsilon_0 = m^2 v_n^2 r_n = L_n^2.
  \]

- Now we use the quantization of angular momentum: \( L_n = m_e v_n r_n = n\hbar \).

- Solve for \( r_n \):
  \[
  r_n = n^2 \hbar^2 \varepsilon_0 / \pi me^2 \equiv n^2 a_0.
  \]

- The "Bohr radius" is \( a_0 = 0.053 \text{ nm} = 53 \text{ pm} \).
Bohr Atom Energy Levels

• Solving for the potential energy in the n’th orbit gives:
  \[ U_n = -\frac{e^2}{4\pi\varepsilon_0 r_n} = -\frac{m_e e^4}{4\varepsilon_0^2 n^2 h^2}. \]

• The kinetic energy is
  \[ K_n = \frac{p^2}{2m} = \frac{(n \hbar/r_n)^2}{2m} = \frac{m_e e^4}{8\varepsilon_0^2 n^2 h^2}. \]

• We see that \( K_n = -\frac{1}{2} U_n \) so the total energy of the hydrogen atom is
  \[ E_n = K_n + U_n = -\frac{m_e e^4}{8\varepsilon_0^2 n^2 h^2}. \]

• Plugging in numbers for the constants, we obtain
  \[ E_n = -13.6 \text{ eV} / n^2. \]
Bohr Model Transition Energies

(a) “Permitted” orbits of an electron in the Bohr model of a hydrogen atom (not to scale). Arrows indicate the transitions responsible for some of the lines of various series.

(b) Energy-level diagram for hydrogen, showing some transitions corresponding to the various series.

- Lyman series
- Paschen series
- Brackett series
- Pfund series

<table>
<thead>
<tr>
<th>n</th>
<th>Lyman series</th>
<th>Paschen series</th>
<th>Brackett series</th>
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<tbody>
<tr>
<td>7</td>
<td>-0.28 eV</td>
<td>-0.38 eV</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>-0.54 eV</td>
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<td>5</td>
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<tr>
<td>n = 1</td>
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<tr>
<td></td>
<td>-3.40 eV</td>
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<tr>
<td></td>
<td>-13.6 eV</td>
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Rydberg Formula for Hydrogen

With $E_n = \frac{-13.6 \text{ eV}}{n^2}$, the Bohr model of the hydrogen atom predicts that a transition from an (upper) level “m” to (lower) level “n” will result in emission of a photon of energy

$$E = 13.6 \text{ eV} \left( \frac{1}{n^2} - \frac{1}{m^2} \right).$$

Despite the simplicity of Bohr’s model, this is exactly (well, almost…) what is observed:

Light emitted by hydrogen gas discharge: Balmer Series

All Balmer lines beyond $H_\delta$ are in the ultraviolet spectrum.

$H_\alpha, H_\beta, H_\gamma,$ and $H_\delta$ are in the visible region of the spectrum.
Reduced Mass

• Our result $E_n = -\frac{m_e e^4}{8\varepsilon_0^2 n^2 h^2}$ is not quite right. Why?

• In our derivation we assumed that the nucleus is at rest, corresponding to infinite mass. However, since the mass of the proton is large compared with the electron, but not infinite, the proton and electron each orbit their common center of mass.

• The correction is to replace the electron mass in the Rydberg formula with a reduced mass:

$$\frac{1}{m_r} = \frac{1}{m_e} + \frac{1}{m_p}$$

$$m_r = 0.99946 \ m_e.$$  

• It is clear that this is a small effect. Yet, the spectral lines of the "heavy" isotope of hydrogen deuterium (one proton and one neutron in the nucleus) are shifted by 0.027% compared to the lines of hydrogen. This is how deuterium was discovered.
Bohr Model Takeaway

The Bohr atom is based on E&M plus the assumption that circular orbits have an integral number of wavelengths. It leads to:

- Quantized radii
- Quantized angular momenta
- Quantized energies

The energy levels are predicted correctly, and the concept of angular momentum quantization is also correct.

However, the model predicts that the n’th energy level always has an angular momentum of n\hbar, which is not confirmed experimentally.

Thus, the Bohr model allows us to start understanding quantization, but it is not really right. Full understanding requires the theory of quantum mechanics.
For a proton to have the same wavelength as an electron:

- a) it has to have the same momentum as the electron
- b) it has to have the same kinetic energy as the electron
- c) it has to have the same mass as the electron
- d) it has to have the same speed as the electron
- e) A proton can never have the same wavelength as an electron.