Physics 228

Ladder in Garage Paradox
Lorentz Transformation of Velocity
L.T. and Simultaneity,
L.T. and Time Dilation, Length Contraction
Doppler Effect
Ladder in Garage Paradox

Ladder won’t fit into garage! What to do?

If the ladder is shoved in fast enough, it will (momentarily) fit due to length contraction!

Yet, from the point of view of the ladder, the garage is moving and is contracted. So the ladder doesn’t fit after all! What gives?
a) The ladder fits in the garage due to ladder’s length contraction.

b) The ladder does not fit in the garage because of contraction of garage.

c) Ladders will never move fast enough for this to work.

d) The ladder fits into the garage as seen in the garage’s frame of reference, but not in ladder’s frame.

e) This is a logical contradiction that disproves special relativity.
Ladder in Garage Paradox

Stationary garage, moving ladder:
Doors close simultaneously!

Stationary ladder, moving garage:
Doors close one after the other!
Review: Lorentz Transformation

\[ x' = \gamma (x - u \, \tau') \]
\[ y' = y \]
\[ z' = z \]
\[ \tau' = \gamma (\tau - u \, x/c^2) \]

\[ x = \gamma (x' + u \, \tau') \]
\[ y = y' \]
\[ z = z' \]
\[ \tau = \gamma (\tau' + u \, x'/c^2) \]

Mavis moving at \( u = 0.6 \, c \)

(lines of simultaneity are horizontal in Stanley’s frame)
\[ x' = \gamma (x - u t) \]
\[ y' = \gamma \]
\[ z' = z \]
\[ t' = \gamma (t - u x/c^2) \]

\[ x = \gamma (x' + u t') \]
\[ y = y' \]
\[ z = z' \]
\[ t = \gamma (t' + u x'/c^2) \]

When constructing this space-time diagram, how would we find Mavis's world line?

a) By setting \( x = 0 \) in the LT.
b) By setting \( t = 0 \) in the LT.
c) By setting \( x' = 0 \) in the LT.
d) By setting \( t' = 0 \) in the LT.
e) None of the above.
In nonrelativistic physics, velocities just add. In relativity, this can’t be the case, otherwise a light beam that was emitted by a moving source would move faster than c. How do we transform velocities?

Let’s say we have a bird flying at velocity $v$ in Stanley’s frame, corresponding to velocity $v'$ in Mavis’s frame. (As always, Mavis is moving with velocity $u$ with respect to Stanley.)

The transverse $(y, z)$ components of velocity are, of course, the same in the two frames. To transform the $x$-component, let’s pick 2 events on the world line of the bird, and transform them into Mavis’ frame:
Lorentz Transform of Velocity

In Stanley’s frame the space interval between events A and B is $\Delta x$, and the time interval is $\Delta t$.

In Mavis’s frame, those intervals are $\Delta x'$ and $\Delta t'$, respectively:

$$
\begin{align*}
  x'_A &= \gamma (x_A - u t_A) \\
  x'_B &= \gamma (x_B - u t_B) \\
  \end{align*}
$$

$$
\begin{align*}
  (x'_A - x'_B) &= \gamma [(x_A - x_B) - u (t_A - t_B)] \\
  \Delta x' &= \gamma (\Delta x - u \Delta t) \\
  \Delta t' &= \gamma (\Delta t - u \Delta x/c^2) \\
  v' &= \Delta x' / \Delta t' = (\Delta x - u \Delta t) / (\Delta t - u \Delta x/c^2)
\end{align*}
$$

Now divide through by $\Delta t$, and recall that $\Delta x / \Delta t = v$:

$$
\begin{align*}
  v' &= (v - u) / (1 - uv/c^2) \quad \text{(Lorentz velocity transformation)} \\
  v &= (v' + u) / (1 + uv'/c^2)
\end{align*}
$$
Lorentz Transform and Simultaneity

Consider two events that are simultaneous in Stanley’s frame:
Event 1 at \( x = y = z = t = 0 \), and event 2 at \( x = L, y = z = t = 0 \).
Are they simultaneous in Mavis’ frame?

Apply Lorentz Transformation:
\[
\begin{align*}
x' &= \gamma (x - ut) \\
y' &= y \\
z' &= z \\
t' &= \gamma (t - ux/c^2)
\end{align*}
\]
We obtain for event 1 that \( x' = y' = z' = t' = 0 \).
We obtain for event 2 that \( y' = z' = 0 \), but \( x' = \gamma L \), and \( t' = -\gamma u L/c^2 \).

In Stanley’s frame, the events are simultaneous but a distance \( L \) apart. In Mavis’ frame, the events are separated in space as well as in time!
Lorentz Transform and Time Dilation

Is time dilation consistent with the Lorentz transform?

Consider two events at the same position in Mavis’ frame, at different times (e.g., Mavis’ heart beating twice):
event 1 at \( x' = y' = z' = t' = 0 \), and event 2 at \( x' = y' = z' = 0, \ t' = T \).

What are the coordinates in Stanley’s frame?

Apply Lorentz Transformation:
\[
x = \gamma(x' + u t') \quad y = y' \quad z = z' \quad t = \gamma(t' + u x'/c^2)
\]

We obtain for event 1 that \( x = y = z = t = 0 \).

We obtain for event 2 that \( y = z = 0 \), but \( x = \gamma u T \), and \( t = \gamma T \).

In Mavis’ frame, the events are at the same place, separated by a time \( T \).

In Stanley’s frame, more time (\( \gamma T \)) passes between the events, and the events happen a distance \( \gamma u T \) apart.
Lorentz Transform and Length Contraction

Is length contraction consistent with the Lorentz transform?

Consider two ends of a ruler held by Mavis on the train. The rear end is at \( x_r' = 0 \), and the front end at \( x_f' = L_0 \) (time independent).

What is the distance between the ends measured simultaneously in Stanley's frame?

\[
0 = x_r' = \gamma (x_r - ut) \quad \text{(position of rear end at any given time t)}
\]

\[
L_0 = x_f' = \gamma (x_f - ut) \quad \text{(position of front end at time t)}
\]

Take difference: \( L_0 = \gamma (x_f - x_r) \)

The Length measured by Stanley is \( L = x_f - x_r = L_0 / \gamma \) (measure both ends at the same time).
Doppler Effect

In the train frame, the light source has frequency $f_0$ and period $T_0$. In Stanley’s frame the waves are emitted a time $T = \gamma T_0$ apart (time dilation).

But what is the time interval at which they are received by Stanley?

From the diagram, we see $\lambda = cT - uT = T(c - u)$.

The frequency at which the wave crests reach Stanley is

$$f = \frac{c}{\lambda} = \frac{c}{T(c - u)} = \frac{1}{\gamma T_0 (1 - \beta)}$$

$$= f_0 \sqrt{\frac{1 - \beta^2}{1 - \beta}} = f_0 \sqrt{\frac{1 + \beta}{1 - \beta}} = f_0 \sqrt{\frac{c + u}{c - u}}$$
Doppler Effect

Stanley perceives an increased frequency given by:

\[ f = f_0 \sqrt{\frac{c+u}{c-u}} \]  (approaching source: “blue shift”)

If the source were receding from the observer at speed \( u \):

\[ f = f_0 \sqrt{\frac{c-u}{c+u}} \]  (receding source: “red shift”)

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Doppler Effect for Space Twins

space trip starts at birth
\[ u = 0.6 \, c \]
\[ \gamma = 1.25 \]

Yearly birthday greetings arrive at 2 year intervals (red shift), or at \( \frac{1}{2} \) year intervals (blue shift)
You are moving toward a mirror at speed $u$. You shine a laser beam into the mirror with light frequency $f$. You observe the reflected light to have frequency

\[ a) \quad f' = f \]
\[ b) \quad f' = f \sqrt{\frac{c+u}{c-u}} \]
\[ c) \quad f' = f \sqrt{\frac{c-u}{c+u}} \]
\[ d) \quad f' = f \frac{c+u}{c-u} \]
\[ e) \quad f' = f \frac{c-u}{c+u} \]
Doppler Effect

Note that in the Doppler effect formulas, only the **relative velocity** between source and observer matters (as it must be according to the principle of relativity).

This is different from the Doppler effect for sound waves, where we must distinguish between moving source/stationary observer and vice versa.
Why does the principle of relativity (i.e., there is no “special” frame of reference) not apply to sound waves?

a) Sound waves travel much slower than the speed of light.

b) Sound has nothing to do with relativity.

c) Sound waves have by their very nature a special frame of reference: The medium (air) they travel in. This allows us to decide who is moving and who is stationary.

d) The principle of relativity applies, and the Doppler shift formula derived for light must also apply to sound.

e) The formulas for sound also apply to light.
Applications of Doppler Effect

- In Astronomy, find out how celestial objects move with respect to us.
- Can also measure the temperature of light-emitting or light-absorbing materials (gases or plasma).

- In Meteorology: Find out how the rain clouds move (identify tornadoes before they touch down!) “Doppler radar”

- Law enforcement: catch speeders with “radar gun”.

- Medicine: Measure how fast blood is flowing inside body (use Doppler ultrasound, record “echocardiogram”).
Applications of Doppler Effect

sonogram movie of beating heart

Doppler echocardiogram movie
Lorentz Transformations vs. Rotations

The Lorentz transform is similar to a rotation of a regular vector (a “3-vector”):

\[
\begin{align*}
    x' &= \cos(\theta) x + \sin(\theta) y \\
    y' &= -\sin(\theta) x + \cos(\theta) y \\
    z' &= z
\end{align*}
\]

Some coordinates are mixed together, others are unaffected, and the length of a vector is the same in both coordinate systems (“invariant”).

The Lorentz transform is analogous: we “rotate” space-time, mixing space and time!

\[
\begin{align*}
    x' &= \gamma (x - u t) \\
    y' &= y \\
    z' &= z \\
    t' &= \gamma (t - \frac{u x}{c^2})
\end{align*}
\]
4-Vectors and Lorentz Invariants

The Lorentz transform is analogous: we “rotate” space-time, mixing space and time!

The 4-coordinate object being transformed \((ct, x, y, z)\) is called a “Lorentz vector” or “4-vector”.

The 4-vector’s proper length is \(s\) (called a “space-time interval”), and its square \(s^2 = (ct)^2 - x^2 - y^2 - z^2\), remain unchanged under this transformation.

Such quantities, which are the same for all observers (in any inertial frames) are called “Lorentz invariants”, or “relativistic invariants”.