Physics 228

Special Relativity:
Lorentz Transformation
Space-Time Diagrams
Twin Paradox Revisited
Review: Galilean Transformation

Frame $S'$ moves relative to frame $S$ with constant velocity $u$ along the common $x$-$x'$-axis.

Origins $O$ and $O'$ coincide at time $t = 0 = t'$.

It the primed frame moves at velocity $u$ in the $x$-direction, in non-relativistic physics the coordinates of an event in the two frames are related by:

\[
\begin{align*}
x' &= x - ut \\
y' &= y \\
z' &= z \\
t' &= t
\end{align*}
\]

\[
\begin{align*}
x &= x' + ut \\
y &= y' \\
z &= z' \\
t &= t'
\end{align*}
\]

Valid for small velocities $u$ only!
Lorentz Transformation

In special relativity, the coordinate transformation becomes:

\[
\begin{align*}
x' &= \gamma (x - u t) \\
y' &= y \\
z' &= z \\
t' &= \gamma (t - u x/c^2)
\end{align*}
\]

\[
\begin{align*}
x &= \gamma (x' + u t') \\
y &= y' \\
z &= z' \\
t &= \gamma (t' + u x'/c^2)
\end{align*}
\]

Frame \(S'\) moves relative to frame \(S\) with constant velocity \(u\) along the common \(x-x'\)-axis.

Origins \(O\) and \(O'\) coincide at time \(t = 0 = t'\).

The Lorentz coordinate transformation relates the spacetime coordinates of an event as measured in the two frames: \((x, y, z, t)\) in frame \(S\) and \((x', y', z', t')\) in frame \(S'\).
Space-Time Diagrams

Simplified Space-Time Diagram for Particle Moving at Constant Speed

Simplified Space-Time Diagram Normalizes Axis Scales to Seconds and Light-Seconds

Space-Time Diagram showing Possible World Lines

Light Cone With Two Space Dimensions

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Clicker Question

Which spacetime diagram is most appropriate for an ideal pendulum?

A)  

B)  

C)  

D)
Starting from event 1, which other events could be reached by travel?

a) A  
b) B  
c) C  
d) More than one  
e) None
Clocks in Space

• To simplify the discussion, we are going to pretend that space is filled with an infinite array of clocks, all synchronized with each other.

• Since time is relative to the observer, we need two sets of clocks: One fixed to Stanley’s frame, and one traveling with Mavis in her frame.

• As we have seen, the two sets of clocks will show different times.

• In particular, Stanley will observe Mavis’s clocks as not synchronized, and vice versa.

• In this way, any “event” can be assigned definite \( x, y, z, \) and \( t \) coordinates, as well as definite \( x', y', z', \) and \( t' \) coordinates.

The grid is three dimensional; identical planes of clocks lie in front of and behind the page, connected by grid lines perpendicular to the page.

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Space-Time Diagrams of Clocks in Space

Stanley’s clocks

Mavis’s clocks

No, they’re not!

All of my clocks are synchronized.

Mavis

Stanley

Stanley’s clocks

Mavis’s clocks
Space-Time Diagram of Moving Frame (Galilei Transformation)

\[ x' = x - u \cdot t \]
\[ y' = y \]
\[ z' = z \]
\[ t' = t \]

\[ x = x' + u \cdot t' \]
\[ y = y' \]
\[ z = z' \]
\[ t = t' \]
Space-Time Diagram of Moving Frame (Lorentz Transformation)

\[ x' = \gamma (x - ut) \]
\[ y' = y \]
\[ z' = z \]
\[ t' = \gamma (t - ux/c^2) \]

\[ x = \gamma (x' + ut') \]
\[ y = y' \]
\[ z = z' \]
\[ t = \gamma (t' + u x'/c^2) \]

Mavis moving at \( u = 0.6c \)

(lines of simultaneity are horizontal in Stanley’s frame)
Continuous Change of Reference Frame
(Animation)

Mavis’s frame (moving at variable velocity $v$), as depicted by Stanley. Events A, B, C are simultaneous in Stanley’s frame, but not in Mavis’ frame.
Clicker Question

Which line could represent a set of synchronized clocks in the spaceship frame?

a) AB  
b) CD  
c) EF
As one of Mavis’s clocks zooms past Stanley’s synchronized clock array, we compare readings every time Mavis’s clock passes one of Stanley’s. We conclude Mavis’s clock runs slow.

Thus, from Stanley’s point of view, Mavis’s clock runs slow.

As one of Stanley’s clocks zooms past Mavis’s synchronized clock array, we compare readings every time Stanley’s clock passes one of Mavis’s. We conclude Stanley’s clock runs slow.

Thus, from Mavis’s point of view, Stanley’s clock runs slow.
Mavis is traveling at relativistic speed on a very long train. There are accurate clocks on each car. All the clocks on the train are synchronized with each other, in the train's frame. Stanley is standing on the platform as the train whizzes by, holding a clock in his hand. Every time one of the train's clocks passes by Stanley and his clock, he snaps a picture of both clocks next to each other. When comparing the clock readings,

A. Stanley and Mavis agree that Stanley's clock advances faster in comparison to the readings of Mavis' clocks, since Mavis' moving clocks are slowed by time dilation.

B. Stanley and Mavis agree that Stanley's clock advances more slowly in comparison to the readings of Mavis' clocks, since his clock is moving with respect to the train, and the measurements are taken in the train's reference frame (the synchronized clocks).

C. Stanley and Mavis agree that Stanley's clock advances at the same rate as the readings on Mavis' clocks.

D. It depends on who you ask: Stanley concludes that his clock runs fast, while Mavis concludes Stanley's clock to run slow, compared to the readings on Mavis' clocks.

E. This is an unsolvable paradox.
NASA tests twin paradox!

Mark Kelly stayed on Earth.

Scott Kelly went to the space station, orbiting at 4.7 mi/s, for one year. When he returned, he was 9 milliseconds younger!
Twin Paradox Revisited

- ct
- traveling twin
- stationary twin
- simultaneity planes (ret. trip)
- simultaneity planes (trip out)

Al

Al

Bert

Bert
Twin Paradox Revisited

Space trip starts at birth
\[ u = 0.6 \, c \]
\[ \gamma = 1.25 \]

World line of space twin

World line of earth twin

Lines of simultaneity before turnaround

Lines of simultaneity after turnaround

Age of earth twin

Distance (lightyears)

Time (years)
Signaling Between Twins

Space trip starts at birth.
\[ u = 0.6 \, c \]
\[ \gamma = 1.25 \]

Each year on his birthday, each twin sends the other a snapchat greeting.

Birthday greetings travel at the speed of light (45 degree diagonal lines).
Ladder in Garage Paradox

Ladder won’t fit into garage! What to do?

If the ladder is shoved in fast enough, it will (momentarily) fit due to length contraction!

Yet, from the point of view of the ladder, the garage is moving and is contracted. So the ladder doesn’t fit after all! What gives?
a) The ladder fits in the garage due to ladder’s length contraction.

b) The ladder does not fit in the garage because of contraction of garage.

c) Ladders will never move fast enough for this to work.

d) The ladder fits into the garage as seen in the garage’s frame of reference, but not in ladder’s frame.

e) This is a logical contradiction that disproves special relativity.
Ladder in Garage Paradox

Stationary garage, moving ladder:
Doors close simultaneously!

Stationary ladder, moving garage:
Doors close one after the other!
When constructing this space-time diagram, how would we find Mavis’s world line?

a) By setting $x = 0$ in the LT.

b) By setting $t = 0$ in the LT.

c) By setting $x' = 0$ in the LT.

d) By setting $t' = 0$ in the LT.

e) None of the above.