Physics 228

Twin Paradox in Detail
Lorentz Transformation of Velocity
L.T. and Simultaneity,
L.T. and Time Dilation, Length Contraction
Doppler Effect
Time Dilation and Principle of Relativity

As one of Mavis’s clocks zooms past Stanley’s synchronized clock array, we compare readings every time Mavis’s clock passes one of Stanley’s. We conclude Mavis’s clock runs slow.

Thus, from Stanley’s point of view, Mavis’s clock runs slow.

As one of Stanley’s clocks zooms past Mavis’s synchronized clock array, we compare readings every time Stanley’s clock passes one of Mavis’s. We conclude Stanley’s clock runs slow.

Thus, from Mavis’s point of view, Stanley’s clock runs slow.
Twin Paradox Revisited

Al

Bert

stationary twin

traveling twin

simultaneity planes (ret. trip)

simultaneity planes (trip out)

Al

Bert
Twin Paradox Revisited

- Space trip starts at birth
  - $u = 0.6 \, c$
  - $\gamma = 1.25$

**Diagram:***
- **Time (years):**
  - Marked points: 1, 2, 3, 4, 5, 6, 7, 8
- **Distance (lightyears):**
  - Marked points: 0.75, 1.5, 2.25, 3
- **World line of space twin**
- **World line of earth twin**
- **Lines of simultaneity before turnaround**
- **Lines of simultaneity after turnaround**
- **Simultaneity Gap**
- **Age of space twin**
- **Age of earth twin**
Signaling Between Twins

Space trip starts at birth.
\[ u = 0.6 \, c \]
\[ \gamma = 1.25 \]

Each year on his birthday, each twin sends the other a snapchat greeting.

Birthday greetings travel at the speed of light (45 degree diagonal lines).
Lorentz Transform of Velocity

In nonrelativistic physics, velocities just add. In relativity, this can’t be the case, otherwise a light beam that was emitted by a moving source would move faster than c. How do we transform velocities?

Let’s say we have a bird flying at velocity \(v\) in Stanley’s frame, corresponding to velocity \(v’\) in Mavis’s frame. (As always, Mavis is moving with velocity \(u\) with respect to Stanley.)

The transverse \((y, z)\) components of velocity are, of course, the same in the two frames. To transform the \(x\)-component, let’s pick 2 events on the world line of the bird, and transform them into Mavis’ frame:
Lorentz Transform of Velocity

In Stanley’s frame the space interval between events A and B is \( \Delta x \), and the time interval is \( \Delta t \).

In Mavis’s frame, those intervals are \( \Delta x' \) and \( \Delta t' \), respectively:

\[
\begin{align*}
\Delta x' &= \gamma (\Delta x - u \Delta t) \\
\Delta t' &= \gamma (\Delta t - u \Delta x/c^2)
\end{align*}
\]

\[
v' = \Delta x'/ \Delta t' = (\Delta x - u \Delta t) / (\Delta t - u \Delta x/c^2)
\]

Now divide through by \( \Delta t \), and recall that \( \Delta x/ \Delta t = v \):

\[
v' = (v - u) / (1 - uv/c^2) \quad \text{(Lorentz velocity transformation)}
\]

\[
v = (v' + u) / (1 + uv'/c^2)
\]
Lorentz Transform and Simultaneity

Consider two events that are simultaneous in Stanley’s frame: Event 1 at \( x = y = z = t = 0 \), and event 2 at \( x = L, y = z = t = 0 \). Are they simultaneous in Mavis’ frame?

Apply Lorentz Transformation:

\[
\begin{align*}
x' &= \gamma (x - ut) \\
y' &= y \\
z' &= z \\
t' &= \gamma (t - ux/c^2)
\end{align*}
\]

We obtain for event 1 that \( x' = y' = z' = t' = 0 \).

We obtain for event 2 that \( y' = z' = 0 \), but \( x' = \gamma L \), and \( t' = -\gamma u L/c^2 \).

In Stanley’s frame, the events are simultaneous but a distance \( L \) apart. In Mavis’ frame, the events are separated in space as well as in time!
Lorentz Transform and Time Dilation

Is time dilation consistent with the Lorentz transform?

Consider two events at the same position in Mavis’ frame, at different times (e.g., Mavis’ heart beating twice):
event 1 at $x'=y'=z'=t'=0$, and event 2 at $x'=y'=z'=0$, $t'=T$.

What are the coordinates in Stanley’s frame?

Apply Lorentz Transformation:

$x = \gamma(x' + ut')$ \quad $y = y'$ \quad $z = z'$ \quad $t = \gamma(t' + ux'/c^2)$

We obtain for event 1 that $x = y = z = t = 0$.

We obtain for event 2 that $y = z = 0$, but $x = \gamma u T$, and $t = \gamma T$.

In Mavis’ frame, the events are at the same place, separated by a time $T$.

In Stanley’s frame, more time ($\gamma T$) passes between the events, and the events happen a distance $\gamma u T$ apart.
Lorentz Transform and Length Contraction

Is length contraction consistent with the Lorentz transform?

Consider two ends of a ruler held by Mavis on the train. The rear end is at $x_r' = 0$, and the front end at $x_f' = L_0$ (time independent).

What is the distance between the ends measured simultaneously in Stanley's frame?

\[ 0 = x_r' = \gamma (x_r - ut) \quad \text{(position of rear end at any given time t)} \]
\[ L_0 = x_f' = \gamma (x_f - ut) \quad \text{(position of front end at time t)} \]

Take difference: $L_0 = \gamma (x_f - x_r)$

The Length measured by Stanley is
\[ L = x_f - x_r = L_0 / \gamma \quad \text{(measure both ends at the same time)}. \]
Doppler Effect

In the train frame, the light source has frequency $f_0$ and period $T_0$. In Stanley's frame the waves are emitted a time $T = \gamma T_0$ apart (time dilation).

But what is the time interval at which they are received by Stanley?

From the diagram, we see $\lambda = cT - uT = T(c - u)$.

The frequency at which the wave crests reach Stanley is

$$f = \frac{c}{\lambda} = \frac{c}{T(c - u)} = \frac{1}{\gamma T_0 (1 - \beta)}$$

$$= f_0 \sqrt{\frac{1 - \beta^2}{1 - \beta}} = f_0 \sqrt{\frac{1 + \beta}{1 - \beta}} = f_0 \sqrt{\frac{c + u}{c - u}}$$
Stanley perceives an increased frequency given by:

\[ f = f_0 \sqrt{\frac{c+u}{c-u}} \]  (approaching source: "blue shift")

If the source were receding from the observer at speed \( u \):

\[ f = f_0 \sqrt{\frac{c-u}{c+u}} \]  (receding source: "red shift")
Doppler Effect for Space Twins

- Space trip starts at birth
  - $u = 0.6 \, c$
  - $\gamma = 1.25$

- Yearly birthday greetings arrive at 2 year intervals (red shift), or at ½ year intervals (blue shift)
Doppler Effect

Note that in the Doppler effect formulas, only the relative velocity between source and observer matters (as it must be according to the principle of relativity).

This is different from the Doppler effect for sound waves, where we must distinguish between moving source/stationary observer and vice versa.
Applications of Doppler Effect

- In Astronomy, find out how celestial objects move with respect to us.
- Can also measure the temperature of light-emitting or light-absorbing materials (gases or plasma).

- In Meteorology: Find out how the rain clouds move (identify tornadoes before they touch down!) “Doppler radar”

- Law enforcement: catch speeders with “radar gun”.

- Medicine: Measure how fast blood is flowing inside body (use Doppler ultrasound, record “echocardiogram”).
Applications of Doppler Effect

sonogram movie of beating heart

Doppler echocardiogram movie