Outline:

- Self-induction and self-inductance.
- Inductance of a solenoid.
- The energy of a magnetic field.
- Mutual Inductance.
Gauss’s Law: charges are sources of electric field (and a non-zero net electric field flux through a closed surface); field lines begin and end on charges.

\[
\int_{\text{surf}} \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\varepsilon_0}
\]

No magnetic monopoles; magnetic field lines form closed loops.

\[
\int_{\text{surf}} \vec{B} \cdot d\vec{A} = 0
\]

Faraday’s Law of electromagnetic induction; a time-dependent \( \Phi_B \) generates \( \vec{E} \).

\[
\oint_{\text{loop}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_{\text{surf}} \vec{B} \cdot d\vec{A}
\]

\[
E = -\frac{d\Phi_B}{dt}
\]

Generalized Ampere’s Law; \( \vec{B} \) is produced by both currents and time-dependent \( \Phi_E \).

\[
\oint_{\text{loop}} \vec{B} \cdot d\vec{l} = \mu_0 \int_{\text{surf}} \left( j + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A}
\]

The force on a (moving) charge.

\[
\vec{F} = q \left( \vec{E} + \vec{v} \times \vec{B} \right)
\]
Two identical concentric loops are arranged as shown in the Figure. One loop has a steady current flowing through it (provided by a power supply). When the power is turned off, in what direction does the induced current flow in loop 2?

A. It flows clockwise.
B. It flows counterclockwise.
C. There is no induced current.
D. Something else happens.
E. The answer depends on what direction the current was flowing in loop 1.
Induced E.M.F. and its consequences

1. \( \text{external } \vec{B}(t) \)

2. \( I(t) \)

3. \( I(t) \)

Faraday: \( E = -\frac{d\Phi_B}{dt} \)
Inductance at work: Wireless charging
The flux depends on the current, so a wire loop with a changing current induces an "additional" e.m.f. in itself that opposes the changes in the current and, thus, in the magnetic flux (sometimes called the back e.m.f.).

Since the magnetic flux through the loop is proportional to \( I \), we may write:

\[ \Phi = LI \]

The proportionality constant \( L \) is called the inductance (or self-inductance) of the loop/solenoid, with units \( \text{T m}^2/\text{A} = \text{Henry (H)} \). Its value depends on the geometry of the loop/solenoid.

The back e.m.f. is then:

\[ \mathcal{E} = -\frac{d\Phi}{dt} = -L\frac{dI}{dt} \]

Inductance as a circuit element:
A long solenoid of radius \( r \) and length \( \ell \) with the total number of turns \( N \):

\[
L \equiv \frac{\Phi}{I}
\]

The magnetic field inside the solenoid:

\[
\Phi = B\pi r^2 N = \frac{\mu_0 N^2 I}{\ell} \pi r^2 = \mu_0 n^2 I \ell \pi r^2
\]

\[
L = \frac{\Phi}{I} = \frac{\mu_0 N^2}{\ell} \pi r^2 = \mu_0 n^2 \ell \pi r^2 = \mu_0 n^2 \cdot (\text{volume})
\]

Note that the inductance scales as \( N^2 \): \( B \) is proportional to \( \frac{N}{\ell} \) and the net flux is \( BAN \).

Let’s plug some numbers: \( \ell = 0.1 \text{m}, N = 100, r = 0.01 \text{m} \)

\[
L = \mu_0 n^2 \ell \pi r^2 = 4\pi \cdot 10^{-7} \cdot \left(\frac{100}{0.1}\right)^2 \cdot 0.1 \cdot \pi (0.01)^2 \approx 40 \mu\text{H}
\]
A current $i$ flows through an inductor $L$ in the direction from point $b$ toward point $a$. There is zero resistance in the wires of the inductor. If the current is decreasing,

A. the potential is greater at point $a$ than at point $b$.
B. the potential is less at point $a$ than at point $b$.
C. the answer depends on the magnitude of $\frac{di}{dt}$ compared to the magnitude of $i$.
D. The answer depends on the value of the inductance $L$.
E. both C. and D. are correct.
An inductor affects the current flow only if there is a change in current \( \frac{di}{dt} \neq 0 \). The sign of \( \frac{di}{dt} \) determines the sign of the induced e.m.f.
Energy of Magnetic Field

To increase a current in a solenoid, we have to do some work: **we work against the back e.m.f.**

Let’s ramp up the current $i(t)$ from 0 to the final value $I$:

$$V = -\mathcal{E} = L \frac{di}{dt} \quad P = Vi = L \frac{di}{dt} \quad P \, dt = L \, i \, di$$

$$dU = P(t) \, dt$$

$P$(power) - the rate at which energy is being delivered to the inductor from external sources.

The net work to ramp up the current from 0 to $I$:

$$U = \int_{0}^{I} P(t) \, dt = \int_{0}^{I} Li \cdot di = \frac{1}{2} LI^2$$

This energy $U$ is stored **in the magnetic field** created by the current.

$$L = \mu_0 n^2 \cdot (volume) \quad B = \mu_0 n I$$

$$U_B = \frac{1}{2} \mu_0 n^2 (volume) \cdot I^2 = \frac{B^2}{2\mu_0} \cdot (volume) = u_B \cdot (volume)$$

The energy density of a magnetic field:

$$u_B = \frac{B^2}{2\mu_0} \quad (\text{compare with } u_E = \frac{\varepsilon_0 E^2}{2})$$
Quench of a Superconducting Solenoid

**MRI scanner**: the magnetic field up to 3T within a volume ~ 1 m³.

The magnetic field energy:

\[
U = volume \cdot u_B = 1m^3 \frac{(3T)^2}{2 \cdot 4\pi \cdot 10^{-7} \frac{Wb}{A \cdot m}} \approx 4MJ
\]

The capacity of the LHe dewar: ~2,000 liters. The heat of vaporization of liquid helium: 3 kJ/liter. Thus, ~ 1000 liters will be evaporated during the quench.

Magnet quench: [http://www.youtube.com/watch?v=tKj39eWFs10&feature=related](http://www.youtube.com/watch?v=tKj39eWFs10&feature=related)
Inductance vs. Capacitance

\[ L = \frac{\Phi_B}{I} = \frac{2U_B}{I^2} \]

\[ U_B = \frac{1}{2} LI^2 = \frac{1}{2} \frac{\Phi_B^2}{L} \]

\[ U_B = \frac{B^2}{2\mu_0} \cdot \text{(volume)} = u_B \cdot \text{(volume)} \]

\[ C = \frac{Q}{V} = \frac{2U_E}{V^2} \]

\[ U_E = \frac{1}{2} CV^2 = \frac{1}{2} \frac{Q^2}{C} \]

\[ U_E = \frac{\epsilon_0 E^2}{2} \cdot \text{(volume)} = u_E \cdot \text{(volume)} \]
Let’s go back to the situation of two separate wire loops. A time-dependent current in loop 1 produces a time-dependent magnetic field $B_1$. The magnetic flux is linked to loop 1 as well as loop 2. Faraday’s law: the time dependent flux of $B_1$ induces e.m.f. in both loops.

The e.m.f. in loop 2 due to the time-dependent $I_1$ in loop 1:

$$
\varepsilon_2 = -\frac{d\Phi_{1\rightarrow 2}}{dt} \quad \Phi_{1\rightarrow 2} = M_{1\rightarrow 2}I_1
$$

The flux of $B_1$ in loop 2 is proportional to the current $I_1$ in loop 1. The coefficient of proportionality $M_{1\rightarrow 2}$ (the so-called mutual inductance) is, similar to $L$, a purely geometrical quantity; its calculation requires, in general, complicated integration. Also, it’s possible to show that

$$
M_{1\rightarrow 2} = M_{2\rightarrow 1} = M
$$

$$
\varepsilon_2 = -M \frac{dI_1}{dt} \quad M = \frac{\Phi_{1\rightarrow 2}}{I_1}
$$

Principle of wireless (inductive) phone charger
A steady current flows through an inductor. If the current is doubled while the inductance remains constant, the amount of energy stored in the inductor

A. increases by a factor of $\sqrt{2}$.
B. increases by a factor of 2.
C. increases by a factor of 4.
D. increases by a factor that depends on the geometry of the inductor.
E. none of the above
A small, circular ring of wire (shown in blue) is inside a larger loop of wire that carries a current $I$ as shown. The small ring and the larger loop both lie in the same plane. If $I$ increases, the current that flows in the small ring

A. is clockwise and caused by self-inductance.
B. is counterclockwise and caused by self-inductance.
C. is clockwise and caused by mutual inductance.
D. is counterclockwise and caused by mutual inductance.
E. is zero, since the two rings of wire are not connected.