Lecture 19. Maxwell’s Equations

Outline:

- Ampere’s Law must be modified in dynamics!
- Maxwell’s Fix: Displacement Current.
- Maxwell’s Equations.
In what order do the disks race through a magnetic field?

A. 1, 2, 3, 4  
B. 4, 3, 2, 1  
C. 1, 4, 3, 2  
D. 2, 3, 4, 1  

Ignore changes to the moment of inertia. Assume the B field is constant and the same size as the disks.
Faraday Law Example

Consider a wire loop (radius $a$) and a long solenoid (radius $b$) with a uniform $B(t) = B_0(t/\tau)$ inside and $B = 0$ outside of the solenoid. Find current $I$ in the wire if its resistance is $R$.

\[
\oint_{\text{loop}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_{\text{surf}} \vec{B} \cdot d\vec{A}
\]

\[
2\pi a \cdot E = -\pi b^2 \cdot \frac{dB(t)}{dt} = -\pi b^2 \cdot \frac{B_0}{\tau}
\]

\[
E = -\frac{b^2 B_0}{2a\tau} \quad I = -\frac{b^2 B_0}{2a\tau R}
\]

Note that $B=0$ at the wire’s location!
Still, there is an induced $E \neq 0$. 
The time-dependent $B$ induces EMFs and currents in the walls. The current directions:

- Above the magnet – CCW (looking from the top)
- Below the magnet – CW.
A bar magnet falls through a conducting ring. Which plot of the current vs. time is correct? The positive values of the current correspond to the direction shown in the Figure.

A

B

C

D
What have we got so far?

- Gauss’s Law: Charges are sources of electric field, generating electric field flux through an enclosing surface. Field lines begin and end on charges.

\[
\oint_{\text{surf}} \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\varepsilon_0}
\]

- No magnetic monopoles; magnetic field lines form closed loops.

\[
\oint_{\text{surf}} \vec{B} \cdot d\vec{A} = 0
\]

- Faraday’s Law of electromagnetic induction: Time-dependent \( \Phi_B \) generates \( \vec{E} \).

\[
\oint_{\text{loop}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oint_{\text{surf}} \vec{B} \cdot d\vec{A}
\]

- Ampere’s Law: \( \vec{B} \) is generated by currents.

\[
\oint_{\text{loop}} \vec{B} \cdot d\vec{l} = \mu_0 I
\]

- The force on a moving charge:

\[
\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})
\]
Charging a Capacitor

Consider the capacitor on the right, which is being charged by a current $i_C$

As the charge builds up, the E-field within the capacitor is increasing.

Let’s use Ampere’s law to find the B field around the charging wire:

$$\oint \vec{B}(r) \cdot d\vec{l} = \mu_0 I_{encl}$$

1. Plane surface crossing the wire:
   $$B(r)2\pi r = \mu_0 i_C$$

2. Bulging surface through the capacitor gap:
   $$B(r)2\pi r = 0$$

These can’t both be right. **What gives??**
Maxwell suggested generalization of Ampere’s Law by adding the quantity $\epsilon_0 \frac{\partial \vec{E}}{\partial t}$ to the current density $\vec{J}$.

Define the displacement current density:

$$\vec{J}_D = \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

**Generalized Ampere’s Law:**

$$\oint_{\text{loop}} \vec{B} \cdot d\vec{l} = \mu_0 \int_{\text{surf}} (\vec{J} + \vec{J}_D) \cdot d\vec{A}$$

$$= \mu_0 \int_{\text{surf}} \left( \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A}$$

$$= \mu_0 I_{\text{encl}} + \mu_0 \epsilon_0 \int_{\text{surf}} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A}$$

$$= \mu_0 \left( I_{\text{encl}} + \epsilon_0 \frac{d}{dt} \int_{\text{surf}} \vec{E} \cdot d\vec{A} \right)$$

$$= \mu_0 \left( I_{\text{encl}} + \epsilon_0 \frac{d\Phi_E}{dt} \right)$$
Regardless of the shape of the surface over which we apply Ampere’s law, we now get the same result for $\int_{\text{loop}} \vec{B}(r) \cdot d\vec{l}$:

$$\int_{\text{loop}} \vec{B}(r) \cdot d\vec{l} = \mu_0 \int_{\text{surf}} \left( \vec{j} + \varepsilon_0 \frac{d\vec{E}}{dt} \right) \cdot d\vec{A} = \mu_0 \int_{\text{surf}} \varepsilon_0 \frac{d\vec{E}}{dt} \cdot d\vec{A}$$

$$B(r)2\pi r = \mu_0 \varepsilon_0 \int_{\text{surf}} \frac{d\vec{E}}{dt} \cdot d\vec{A} = \mu_0 \varepsilon_0 \frac{I}{\varepsilon_0 A} A = \mu_0 I$$

The same as for the plane surface intersecting the current-carrying wire!

$$\frac{d\vec{E}}{dt} = \frac{1}{d} \frac{dV}{dt} = \frac{1}{Cd} \frac{dQ}{dt} = \frac{I}{\varepsilon_0 A}$$

$$\frac{dQ}{dt} = I$$

$$C = \varepsilon_0 A/d$$
Maxwell’s Equations

- **Gauss’s Law**: Charges are sources of electric field, generating electric field flux through an enclosing surface. Field lines begin and end on charges.

  \[ \oint_{\text{surf}} \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{encl}}}{\varepsilon_0} \]

- No magnetic monopoles. Magnetic field lines form closed loops.

  \[ \oint_{\text{surf}} \mathbf{B} \cdot d\mathbf{A} = 0 \]

- **Faraday’s Law of electromagnetic induction**: Time-dependent \( \Phi_B \) generates \( E \).

  \[ \oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_{\text{surf}} \mathbf{B} \cdot d\mathbf{A} \]

- **Generalized Ampere’s Law**: \( B \) is produced by both currents and time-dependent \( \Phi_E \).

  \[ \oint_{\text{loop}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \left( j + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{A} \]

- The force on a moving charge (not considered one of Maxwell’s Equations, but just as important):

  \[ \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]
Maxwell’s Equations in Vacuum

\[
\begin{align*}
\oint_{\text{surf}} \mathbf{E} \cdot d\mathbf{A} &= 0 \\
\oint_{\text{surf}} \mathbf{B} \cdot d\mathbf{A} &= 0 \\
\oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} &= -\frac{d}{dt} \int_{\text{surf}} \mathbf{B} \cdot d\mathbf{A} \\
\oint_{\text{loop}} \mathbf{B} \cdot d\mathbf{l} &= \varepsilon_0 \mu_0 \int_{\text{surf}} \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{A}
\end{align*}
\]

Do these equations have non-trivial \((E \neq 0, B \neq 0)\) solutions in vacuum?
Yes, these solutions describe electromagnetic waves propagating at the speed of light!

**Maxwell’s triumph:**
- Prediction of electromagnetic waves that travel through empty space at the speed of light.
- Identification of light as electromagnetic waves.
Plane EM Waves in Vacuum

Generation of EM waves: by *accelerated charges* and, thus, by *AC currents*.

The simplest case is a *plane wave*:

\[
\oint_{\text{surf}} \vec{E} \cdot d\vec{A} = 0 \quad \oint_{\text{surf}} \vec{B} \cdot d\vec{A} = 0
\]

\[
\int_{\text{loop}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_{\text{surf}} \vec{B} \cdot d\vec{A}
\]

\[
\int_{\text{loop}} \vec{B} \cdot d\vec{l} = \varepsilon_0 \mu_0 \int_{\text{surf}} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A}
\]

Derive *wave equations* for both $E$ and $B$ fields:

\[
\frac{\partial^2 E(x, t)}{\partial x^2} - \varepsilon_0 \mu_0 \frac{\partial^2 E(x, t)}{\partial t^2} = 0
\]

\[
\frac{\partial^2 B(x, t)}{\partial x^2} - \varepsilon_0 \mu_0 \frac{\partial^2 B(x, t)}{\partial t^2} = 0
\]
1D Wave Equation

\[ \frac{\partial^2 E(x, t)}{\partial x^2} - \epsilon_0 \mu_0 \frac{\partial^2 E(x, t)}{\partial t^2} = 0 \quad \leftrightarrow \quad \frac{\partial^2 f(x, t)}{\partial x^2} - \alpha \frac{\partial^2 f(x, t)}{\partial t^2} = 0 \]

Units of \( \alpha \): \( \frac{s^2}{m^2} \Rightarrow 1/v^2 \)

\[ \frac{\partial^2 f(x, t)}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 f(x, t)}{\partial t^2} = 0 \]

General wave equation in 1D

Solutions: \( f(x, t) = f(x - vt) \) or \( f(x, t) = f(x + vt) \)

Motion of features of \( f(x, t) \) (crests/troughs): \( x = \pm vt \)

A cosine wave \( f(x, t) = A \cos(kx - \omega t) \) is of this form:

\[ \cos(kx - \omega t) = \cos\left(k\left(x - \frac{\omega}{k} t\right)\right) = \cos(k(x - vt)) \]

where \( v = \frac{\omega}{k} \) is the wave propagation velocity, \( \omega = \frac{2\pi}{T} \) is the angular frequency, and \( k = \frac{2\pi}{\lambda} \) is the wave vector. (\( T \) is the period, \( \lambda \) is the wavelength.)
Electromagnetic Waves in Vacuum

\[ \vec{E}(x, t) = E_0 \cos(kx - \omega t) \hat{j} \]
\[ \vec{B}(x, t) = B_0 \cos(kx - \omega t) \hat{k} \]

\( kx - \omega t \) is the “phase” of the wave.
\( kx - \omega t = \text{const.: A phase front (a surface of constant phase)} \)

**1. In vacuum, EM waves travel at the speed of light** \( c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \).

\[
\frac{\partial^2 E(x, t)}{\partial x^2} - \varepsilon_0 \mu_0 \frac{\partial^2 E(x, t)}{\partial t^2} = 0 \quad \iff \quad \frac{\partial^2 E(x, t)}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 E(x, t)}{\partial t^2} = 0
\]

\[ c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \quad c \approx 3 \cdot 10^8 \text{ m/s} \]

**2. EM waves in free space are **transverse waves.**

\[ \vec{E} \times \vec{B} = \frac{E^2}{\omega} \hat{k} \]
Which of the following is FALSE?

a) Maxwell’s displacement current must be added to the ordinary current in Ampere’s law to make the law complete.

b) Time-varying electric fields give rise to magnetic fields, and time-varying magnetic fields give rise to electric fields.

c) The magnetic flux through a closed surface is always zero.

d) Maxwell’s displacement current is necessary for the existence of electromagnetic waves.

e) Maxwell’s displacement current exists only where a magnetic field changes with time.
Maxwell’s Equations

**Electrostatics**
Electric fields generated by charges at rest

\[ \oint E \cdot dA = \frac{Q}{\varepsilon_0} \]
\[ \oint B \cdot dA = 0 \]
\[ \int E \cdot ds = 0 \]
\[ \int B \cdot ds = \mu_0 I \]

**Magnetostatics**
Magnetic fields generated by time-independent currents

**Electromagnetism**
\[ \frac{d}{dt} \neq 0 \]
Next time: Review for Midterm II. Please email me your questions!