Lecture 3. Electric Field Flux, Gauss’ Law

Last Lecture: Electric Field Lines
Charged particles are fixed on grids having the same spacing. Each charge has the same magnitude $Q$ with signs given in the figure. Rank the magnitude of the electric field (from strong to weak) at the location marked with an “x”.

A. $1 > 2 > 3 > 4$

B. $2 > 3 > 1 = 4$

C. $1 = 4 > 3 > 2$

D. $3 > 2 > 1 > 4$
An electron is released from rest in an electric field. At this moment, the **acceleration** of the electron

A. is in the direction of the electric field at the position of the electron.

B. is directly opposite the direction of the electric field at the position of the electron.

C. is perpendicular to the direction of the electric field at the position of the electron.

D. is zero.

E. not enough information given to decide
(Laminar) flow of fluid: the speed of water, \( v \), and the pipe cross section, \( A \), are known. Calculate how much water you added to the bucket in 1 sec.

\[
\Delta V = A \cdot v \cdot 1s
\]

length (water displacement in 1s)

\[
\Delta V = (\vec{A} \cdot \vec{v}) \cdot 1s
\]

scalar product, \( A \cdot v \cdot \cos \phi \)
Electric Flux for Uniform Field

- Consider a flat area perpendicular to a uniform electric field.

- Increasing the area means that more electric field lines pass through the area, increasing the flux.

- A stronger field means more closely spaced lines, and therefore more flux.
Electric Flux for Uniform Field

- If the area is edge-on to the field, then the area is perpendicular to the field and the flux is zero.

Surface is edge-on to electric field:
• $\vec{E}$ and $\vec{A}$ are perpendicular (the angle between $\vec{E}$ and $\vec{A}$ is $\phi = 90^\circ$).
• The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA \cos 90^\circ = 0$. 

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Electric Flux for Uniform Field

• If the area is not perpendicular to the field, then fewer field lines pass through it.

• In this case the area that counts is the silhouette area that we see when looking in the direction of the field.

Surface is tilted from a face-on orientation by an angle $\phi$:
- The angle between $\vec{E}$ and $\vec{A}$ is $\phi$.
- The flux $\Phi_E = \vec{E} \cdot \vec{A} = EA \cos \phi$. 

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Electric Flux for Non-Uniform Field

- applies to any \textit{vector field} \( \vec{a}(\vec{r}) \) (the velocity field in a flowing fluid, the \textit{electric} field, etc.).

\[ \Phi_E = \int E \cos \phi \, dA = \int E_{\perp} \, dA = \int \vec{E} \cdot d\vec{A} \]

\textbf{Electric flux through a surface}

- Magnitude of electric field \( \vec{E} \)
- Component of \( \vec{E} \) perpendicular to surface
- Angle between \( \vec{E} \) and normal to surface
- Element of surface area
- Vector element of surface area
The flux of electrostatic field:
\[ \Phi_E \rightarrow \frac{N}{C} m^2 \]
\[ (F = qE) \]

The flux of gravitational field:
\[ \Phi_g \rightarrow \frac{N}{kg} m^2 = \frac{m^3}{s^2} \]
\[ (F = mg) \]
Flux through Closed Surfaces

The net flux of the electric field through the closed surface:

$$\Phi_E = \oint \vec{E}(\vec{r}) \cdot d\vec{A}$$

the integral is taken over the whole surface

Convention:
the vector normal to the closed surface points outward.

We integrate over this “Gaussian” surface
At each point of the surface $\vec{E} \parallel d\vec{A}$ ( $\cos \phi = 1$ )

$$\Phi_E = \int \vec{E}(\vec{r}) \cdot d\vec{A} = E(r)4\pi r^2$$

$$= \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} 4\pi r^2 = \frac{q}{\varepsilon_0}$$ - doesn’t depend on $r$

- holds for any vector field whose strength $\propto 1/r^2$. (electrostatic, gravitational)

The flux of the gravitational field of Earth through Earth’s surface?

$$\Phi_g = -g \cdot 4\pi R^2 = -\frac{GM}{R^2} 4\pi R^2 = -4\pi GM$$
What happens if we place the charge outside?

\[ \Phi_E = 0 \]. Every field line entering the surface has to exit again!

The flux magnitudes are the same for these two surfaces.
Gauss’ Law

The total flux of the electric field though any closed surface is proportional to the net charge inside the surface.

\[ \Phi_E \equiv \oint \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{1}{\varepsilon_0} \sum_i q_i \]

The net charge:

\[ Q = \sum_i q_i \quad Q = \int_{volume} \rho(r) \, d\tau \]

Note to mathematicians: The closed surface has to be “oriented” (have an inside and an outside). Thus Gauss’s law doesn’t apply to strange manifolds such as Klein bottles!
Two point charges, $+q$ (in red) and $-q$ (in blue), are arranged as shown.

Through which closed surface(s) is the net electric flux equal to zero?

A. surface $A$  
B. surface $B$  
C. surface $C$  
D. surface $D$  
E. both surface $C$ and surface $D$
Applications of Gauss’ Law

\[ \int \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{Q_{\text{net}}}{\varepsilon_0} \]

- Given the electric field at the surface, Gauss’ Law allows us to find the enclosed charge.
- Given the charge distribution, Gauss’ Law is not usually sufficient for finding *three* components of the vector field \( \mathbf{E} \).
- However, it can be useful to find \( \mathbf{E} \) field whenever the magnitude of the field is constant on a surface and the field’s direction is determined by the symmetry of the charge distribution!

*Useful symmetries:*

- spherical symmetry
- symmetry of infinitely long cylinder
- plane symmetry

A symmetry-adapted choice of a Gaussian surface is crucial!
A sphere of radius $a$ is uniformly charged with a positive charge $Q$. It is surrounded by a uniformly charged shell with charge $-3Q$. In the region between the sphere and the shell ($a < r < b$),

A. the electric field points radially outward.
B. the electric field points radially inward.
C. the electric field points radially outward in parts of the region and radially inward in other parts of the region.
D. the electric field is zero.
E. Not enough information is given to decide.
For which of the following charge distributions would Gauss’s law not be useful for calculating the electric field?

A. a uniformly charged sphere of radius $R$

B. a spherical shell of radius $R$ with charge uniformly distributed over its surface

C. a right circular cylinder of radius $R$ and height $h$ with charge uniformly distributed over its surface

D. an infinitely long circular cylinder of radius $R$ with charge uniformly distributed over its surface

E. Gauss’s law would be useful for finding the electric field in all of these cases.
**Example:** electric field of a uniformly charged ball. Radius $R$, total charge $Q$.

Gaussian surface: a sphere centered at $(\cdot) O$. Its radius $r$ can be smaller or larger than $R$.

Charge density
$$\rho = \frac{Q}{\frac{4}{3} \pi R^3}$$

Enclosed charge
$$E(r) \cdot 4\pi r^2 = \frac{q(r' \leq r)}{\varepsilon_0} = \begin{cases} \frac{\rho \frac{4}{3} \pi r^3}{\varepsilon_0} = \frac{Q}{\varepsilon_0 R^3} & (r < R) \\ \frac{Q}{\varepsilon_0} & (r \geq R) \end{cases}$$

Field outside the sphere is the same as that for a point charge $Q$ at the sphere’s center.

$$\vec{E}(r) = E(r)\hat{r}$$

$$E(r) = \begin{cases} \frac{Qr}{4\pi \varepsilon_0 R^3} & (r < R) \\ \frac{Q}{4\pi \varepsilon_0 r^2} & (r \geq R) \end{cases}$$
**Plane Symmetry**

**Example**: the field of uniformly charged infinite plane.

Gaussian surface: a rectangular box (or cylinder) centered at the plane.

\[ E(x) \cdot 2A = \frac{\sigma A}{\varepsilon_0} \]

enclosed charge

\[ E(x) = \frac{\sigma}{2\varepsilon_0} \]

\[ \vec{E}(x) = \pm E(x)\hat{x} \]

\[ E(x) \]

\[ \frac{\sigma}{2\varepsilon_0} \]

\[ -\frac{\sigma}{2\varepsilon_0} \]

\[ x \]

\[ \sigma \text{ - the surface charge density } [C/m^2] \]
A charge \( q \) is at the center of a cube. What’s the flux of \( E \) through the shaded side?

(A) \( \Phi_E = \frac{q}{\varepsilon_0} \)

(B) \( \Phi_E = \frac{q}{\varepsilon_0} \times \frac{1}{2} \)

(C) \( \Phi_E = \frac{q}{\varepsilon_0} \times \frac{1}{3} \)

(D) \( \Phi_E = \frac{q}{\varepsilon_0} \times \frac{1}{6} \)

(E) \( \Phi_E = \frac{q}{\varepsilon_0} \times \frac{1}{12} \)
**Example 1**: A charge $q$ is at the center of a cube. What’s the flux of $E$ through the shaded side?

$$
\Phi_E = \frac{q}{\varepsilon_0} \times \frac{1}{6}
$$

**Example 2**: A charge $q$ is at the back corner of a cube. What’s the flux of $E$ through the shaded side?

$$
\Phi_E = \frac{q}{\varepsilon_0} \times \frac{1}{24}
$$
Example: the field of uniformly charged infinitely long wire.

Gaussian surface: a cylinder of length $L$ centered at the wire.

$E(r) \cdot 2\pi rl = \frac{\rho l}{\varepsilon_0}$

$E(r) = \frac{\rho}{2\pi \varepsilon_0 r}$

$\mathbf{E}(r) = E(r)\hat{r}$

$r \geq R$

$\rho$ - the linear density of charges (the unit length charge)

$E_{\perp} = E$

$E_{\perp} = 0$
Rank the fluxes $\Phi_i$ through the five surfaces (from most negative to most positive):

\[ A. \quad \Phi_1 < \Phi_2 < \Phi_3 < \Phi_4 < \Phi_5 \]
\[ B. \quad \Phi_1 < \Phi_4 < \Phi_2 < \Phi_3 < \Phi_5 \]
\[ C. \quad \Phi_4 < \Phi_5 < \Phi_2 < \Phi_3 < \Phi_1 \]
\[ D. \quad \Phi_2 < \Phi_4 < \Phi_1 < \Phi_3 < \Phi_5 \]
\[ E. \quad \Phi_5 < \Phi_4 < \Phi_3 < \Phi_2 < \Phi_1 \]