Attention: the list of unregistered iclickers will be posted on our Web page after this lecture.

From the concept of electric field flux – to the calculation of electric fields of complex charge distributions.
(Laminar) flow of fluid: the speed of water, \( v \), and the pipe cross section, \( A \), are known. Calculate how much water you added to the bucket in 1 sec.

\[
\Delta V = A \cdot v \cdot 1s
\]

length (water displacement in 1s)

\[
\Delta V = (\vec{A} \cdot \vec{v}) \cdot 1s
\]

scalar product, \( A \cdot v \cdot \cos \phi \)
- applies to any vector field \( \vec{a}(\vec{r}) \) (the field of velocities in a fluid flow, the \( E \) field, etc.).

The flux of the field \( \vec{a}(\vec{r}) \) through a small element of surface \( d\vec{A} \): 

\[
\Phi_a \equiv \vec{a}(\vec{r}) \cdot d\vec{A} = a(r)A \cdot \cos \phi
\]

scalar!
The flux of electrostatic field:
\[ \Phi_E \rightarrow \frac{N}{C} m^2 \]
\[ (F = qE) \]

The flux of gravitational field:
\[ \Phi_g \rightarrow \frac{N}{kg} m^2 = \frac{m^3}{s^2} \]
\[ (F = mg) \]
The net flux of the electric field through the closed surface:

$$\Phi_E = \oint \vec{E}(\vec{r}) \cdot d\vec{A}$$

The integral is taken over the whole surface.

**Convention:**
the vector normal to the closed surface points **outward**.
Example: Point Charge at the Center of a Spherical Surface

At each point of the surface $\vec{E} \parallel d\vec{A}$ (cos$\phi$=1)

$$\Phi_E = \oint \vec{E}(\vec{r}) \cdot d\vec{A} = E(r)4\pi r^2$$

$$= \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} 4\pi r^2 = \frac{q}{\varepsilon_0}$$

- doesn’t depend on $r$

- holds for any vector field whose strength $\propto 1/r^2$.

(electrostatic, gravitational)

The flux of the gravitational field of Earth through Earth’s surface?

$$\Phi_g = -g \cdot 4\pi R^2 = -G \frac{M}{R^2} 4\pi R^2 = -4\pi GM$$
What happens if we place the charge *outside*? $\Phi_E = 0$

\[ E \text{ field lines} \]

consequence of $1/r^2$ field dependence = continuity of field lines

![Diagram showing field lines around a charge](image)

The flux magnitudes *are the same* for these two surfaces.
Example: gravitational field flux

\[ \Phi_g = -4\pi G m \]

Does the result of flux calculation depends on

1. the shape of the Gaussian surface?
2. The position of the mass within the surface?
3. Presence of the Earth nearby?
Gauss’ Law

The total flux of the electric field though any closed surface is proportional to the net charge inside the surface.

\[ \Phi_E \equiv \oint \mathbf{E}(\mathbf{r}) \cdot d\mathbf{A} = \frac{1}{\varepsilon_0} \sum q_i \]

The net charge:

\[ Q = \sum_i q_i \]

\[ Q = \int_{\text{volume}} \rho(r) \, d\tau \]
Gauss’ Law (cont’d)

- Electrostatics: Gauss’ Law \( \equiv \) Coulomb’s Law.

- Electrodynamics: Gauss’ Law \( \checkmark \) Coulomb’s Law

(the total flux doesn’t depend on the position of charges, so they can move inside the closed surface!)

Gauss’ Law doesn’t imply action-at-a-distance, and we don’t need to modify it in electrodynamics. In fact, this is \textit{the first Maxwell’s equation}. 

\[ \]
Applications of Gauss Law

\[ \int \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{Q_{\text{net}}}{\varepsilon_0} \]

Gauss’ Law (GL) in its integral form (one scalar equation) is not sufficient for finding three components of a vector field \( \mathbf{E} \).

However, GL is very useful whenever it is possible to reduce a 3D (vector) problem to a 1D (scalar) problem. The key is the proper symmetry of a problem.

«Useful» symmetries:

- spherical
- cylindrical + translational
- plane

An intelligent choice of a Gaussian surface is crucial!

Example of an «insufficient» symmetry (cylindrical but without translational): an electric dipole.
**Example**: electric field of a uniformly charged ball. Radius $R$, total charge $Q$.

Gaussian surface: a sphere centered at ($\cdot$) $O$. Its radius $r$ can be smaller or larger than $R$.

\[
\rho = \frac{Q}{\frac{4}{3} \pi R^3}
\]

\[
E(r) \cdot 4\pi r^2 = \frac{q(r' \leq r)}{\varepsilon_0} = \begin{cases} 
\frac{\rho \frac{4}{3} \pi r^3}{\varepsilon_0} = \frac{Q r^3}{\varepsilon_0 R^3} & (r < R) \\
\frac{Q}{\varepsilon_0} & (r \geq R)
\end{cases}
\]

\[
\Phi_E = \int E \cdot ds = \int_{\rho}^{r} E(r') d\rho
\]

\[
\vec{E}(r) = E(r)\hat{r}
\]

\[
E(r) = \begin{cases} 
\frac{Qr}{4\pi\varepsilon_0 R^3} & (r < R) \\
\frac{Q}{4\pi\varepsilon_0 r^2} & (r \geq R)
\end{cases}
\]

The field outside the sphere is the same as that for a point charge $Q$ at the sphere’s center.
Plane Symmetry

**Example**: the field of uniformly charged infinite plane.

Gaussian surface: a rectangular box (or cylinder) centered at the plane.

\[
E(x) \cdot 2A = \frac{\sigma A}{\varepsilon_0}
\]

**Flux**

\[
E(x) = \frac{\sigma}{2\varepsilon_0}
\]

\[
\vec{E}(x) = \pm E(x) \hat{x}
\]

\[\sigma - \text{the surface charge density [C/m}^2\]
Superposition

\[ E_{\text{net}} = E_1 + E_2 \]
Gauss’ Law: works in electrodynamics, in electrostatics it is equivalent to Coulomb’s Law.

Powerful tool for computing the electric fields if a problem is essentially 1D due to symmetry.

Next time: Lecture 4. Applications of Gauss’ Law, Conductors in Electrostatics. §§ 22.5

Read about Metals and Dielectrics.
Appendix I. Cylindrical Symmetry + Translational Symmetry along the axis of symmetry

**Example**: the field of uniformly charged infinitely long wire.

Gaussian surface: a cylinder of length $L$ centered at the wire.

\[ E(r) \cdot 2\pi rl = \frac{\rho l}{\varepsilon_0} \]

Flux

\[ E(r) = \frac{\rho}{2\pi \varepsilon_0 r} \]

Enclosed charge

\[ \vec{E}(r) = E(r) \hat{r} \]

\[ r \geq R \]

$\rho$ - the linear density of charges (the unit length charge)

\[ \propto \frac{1}{r} \]

$R$
Example 1: A charge $q$ is at the center of a cube. What’s the flux of $E$ through the shaded side?

\[ \Phi_E = \frac{q}{\varepsilon_0} \times \frac{1}{6} \]

Example 2: A charge $q$ is at the back corner of a cube. What’s the flux of $E$ through the shaded side?

\[ \Phi_E = \frac{q}{\varepsilon_0} \times \frac{1}{24} \]