Lecture 23. *R-L* and *L-C* Circuits.

**Outline:**

- “Energizing” and “de-energizing” an inductor in *R-L* circuits.
- Oscillations in *L-C-R* circuits.

*The final exam* will be held at CAC GYM / SC 135 on Dec. 19, 4:00 - 7:00 PM.

*Conflict final exam:* No later than on Monday, Dec. 4 students can request the option to take a conflict/makeup final exam in Physics 227. Students should either have an exam at the same time, or at least 3 exams in a 24 hour period. They should contact Prof. Montalvo with details of why they are requesting a conflict exam.
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Initially the switch is closed, the coil $L$ conducts a certain current (depends on the coil’s resistance $r$), and the corresponding magnetic field energy is stored in the coil.

The switch is open at $t = 0$. As the current begins to decrease, the e.m.f. is induced in the coil that opposes the change by “pushing” the current through the coil and the resistor $R$.

\[ E = -L \frac{di(t)}{dt} > 0 \]

$i(t)$ is the instantaneous value of the current through the coil and through the resistor $R$.

Let’s assume that $R \gg r$:

\[ -L \frac{di(t)}{dt} - i(t)R = 0, \quad \frac{di(t)}{dt} + \frac{R}{L}i(t) = 0, \quad \frac{di(t)}{i(t)} = -\frac{R}{L} \frac{dt}{dt} \]

\[ i(t) = I_0 \exp \left( -\frac{t}{L/R} \right) \]

$I_0 = V/r$ - the steady current through the inductor (when switch S was closed).
“De-energizing” an Inductor

\[ i(t) = I_0 \exp \left( -\frac{t}{L/R} \right) \]

\( I_0 = \frac{V}{r} \) - the steady current through the inductor (when switch S was closed).

\[ \tau = \frac{L}{R} \]

- the time constant of the circuit which consists of an inductor \( L \) and resistor \( R \).

Let’s check that \( L/R \) has units of time:

\[ \frac{L}{R} = \frac{LI^2}{RL^2} \rightarrow \frac{J}{J/s} = s \]
$I_0$ was the current through the inductor when the switch was “ON”. In general, $I_0$ is not determined by the net $R$ of the circuit.

The initial rate of energy dissipation: $P \approx I_0^2 R$ (the larger $R$, the faster the energy decreases) – thus, $\tau \propto R$.

Looks similar, right? BUT the current $I_0 = V_0/R$ is inversely proportional to $R$! Thus, the initial rate of energy dissipation: $P \approx V_0^2 / R$ (the larger $R$, the slower energy decreases) – thus, $\tau \propto 1/R$. 
As an inductor opposes any decrease in the current flowing through it, it also opposes any increase in that current.

At \( t = 0 \) the switch is closed \([i(t = 0) = 0]\). As the current begins to increase, the e.m.f. is induced in the coil that opposes the change.

\[
\mathcal{E} - L \frac{di(t)}{dt} - i(t)R = 0
\]

\[
\frac{di(t)}{dt} + \frac{R}{L} i(t) = \frac{\mathcal{E}}{L}, \quad i(t) = \frac{\mathcal{E}}{R} \left[ 1 - \exp \left( -\frac{t}{L/R} \right) \right]
\]

\( t \gg \tau: \quad i(t) = \frac{\mathcal{E}}{R} \)

\[
\tau = \frac{L}{R}
\]
Oscillations in *L-C* Circuits, no external drive

*L-C* circuits: the circuits with TWO elements that can store energy (ideally, without dissipation). The energy flow back and forth between *L* and *C* results in harmonic oscillations of *q(t)* and *i(t)*.

Electric field energy in the capacitor

\[ U_E = \frac{q^2}{2C} \]

Magnetic field energy in the inductor

\[ U_B = \frac{Li^2}{2} \]

Let’s say, at *t* = 0 the capacitor is fully charged, *q* = *q_0*  \( i = 0 \).

\[ \frac{q_0^2}{2C} = \frac{(q(t))^2}{2C} + \frac{L(i(t))^2}{2} \]
Oscillations in L-C Circuits, no external drive (cont’d)

\[-L \frac{di(t)}{dt} - \frac{q(t)}{C} = 0, \quad \frac{di(t)}{dt} \equiv \frac{d^2q(t)}{dt^2}, \quad \frac{d^2q(t)}{dt^2} + \frac{1}{LC} q(t) = 0\]

Solution: \(q(t) = q_0 \cos(\omega t + \varphi_0)\)

- **amplitude**: \(q_0\)
- **angular frequency**: \(\omega\)
- **phase at \(t=0\)**: \(\varphi_0\)

\[i(t) = \frac{dq(t)}{dt} = -\omega q_0 \sin(\omega t + \varphi_0)\]

\[U_E = \frac{q^2}{2C} = \frac{q_0^2}{2C} \cos^2(\omega t) = \frac{q_0^2}{4C} (1 + \cos(2\omega t))\]

\[U_B = \frac{Li^2}{2} = \frac{L\omega^2 q_0^2}{2} \sin^2(\omega t) = \frac{L\omega^2 q_0^2}{4} (1 + \sin(2\omega t))\]
Damping of oscillations in $L$-$C$-$R$ Circuits

\[-L \frac{di(t)}{dt} - \frac{q(t)}{C} - iR = 0, \quad \frac{d^2 q(t)}{dt^2} + \frac{R}{L} \frac{dq(t)}{dt} + \frac{1}{LC} q(t) = 0\]

Solution for weak damping $\frac{1}{LC} \gg \frac{R^2}{4L^2}$:

\[q(t) = q_0 \exp\left(-\frac{Rt}{2L}\right) \cos(\omega^* t + \varphi_0)\]

\[\omega^* = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\]
Next time: Lecture 24. Impedance of AC circuits, §§ 31.3 - 6
Appendix I: “Discharging” an Inductor – energy transformation

Let’s check that the energy stored in the magnetic field is 100% transformed into Joule heat after switch S is open.

The initial magnetic field energy stored in the inductor:

\[ U_B = \frac{1}{2} L (I_0)^2 = \frac{1}{2} L \left(\frac{V}{r}\right)^2 \]

\[ I_0 = \frac{V}{r} \] - the steady current through the inductor (switch S is closed).

Switch S is open at \( t = 0 \). The back e.m.f.:

\[ |\mathcal{E}(t)| = \left| L \frac{di(t)}{dt} \right| = (R + r)i(t) \]

Power dissipated in the resistors:

\[ P = (R + r)i^2(t) \]

\[ i(t) = \frac{V}{r} \exp \left( -\frac{t}{L/(R + r)} \right) \]

\[ \tau = \frac{L}{(R + r)} \]

Net thermal energy released in the resistors:

\[ \int_0^\infty (R + r)i^2(t) dt = (R + r) \int_0^\infty \left(\frac{V}{r}\right)^2 e^{-2t/\tau} dt \]

\[ = (R + r) \left(\frac{V}{r}\right)^2 \left(-\frac{\tau}{2}\right) \left(e^{-\infty/\tau} - e^{-0/\tau}\right) = \frac{1}{2} L \left(\frac{V}{r}\right)^2 \]
Initially the switch is closed, the coil $L$ conducts a certain current, and the corresponding magnetic field energy is stored in the coil. The switch is open at $t = 0$, and the back e.m.f. $\mathcal{E}$ is induced in the coil. Can $|\mathcal{E}(t = 0)|$ be greater than $V$?

A. never

B. always

C. only if $R+r$ is sufficiently small

D. only if $R+r$ is sufficiently large

\[ \mathcal{E} = -L \frac{d}{dt} \left( i_0 \exp \left( -\frac{t}{L/(R+r)} \right) \right) \]
\[ = -L \frac{d}{dt} \left( i_0 \exp \left( -\frac{t}{L/(R+r)} \right) \right) \]
\[ = Li_0 \frac{(R+r)}{L} \exp \left( -\frac{t}{L/(R+r)} \right) \]
\[ = \frac{V}{r} (R+r) \exp \left( -\frac{t}{L/(R+r)} \right) \]
\[ i_0 = \frac{V}{r} \]