Outline:

- Self-induction and self-inductance.
- Inductance of a solenoid.
- The energy of a magnetic field.
- Alternative definition of inductance.
- Mutual Inductance.
Energy Transformations in EM Waves and Circuits

The energy of EM waves traveling in vacuum is stored in both $E$ and $B$ fields (in equal amounts).

Circuit elements that store the $E$ field energy - capacitors.

Circuit elements that store the $B$ field energy – inductors.

By combining capacitors and inductors, we can build the EM oscillators (e.g., generators of EM waves).
Maxwell’s Equations

- Gauss’s Law: charges are sources of electric field (and a non-zero net electric field flux through a closed surface); field lines begin and end on charges.

\[ \oint_{\text{surf}} \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\varepsilon_0} \]

- No magnetic monopoles; magnetic field lines form closed loops.

\[ \oint_{\text{surf}} \vec{B} \cdot d\vec{A} = 0 \]

- Faraday’s Law of electromagnetic induction; a time-dependent \( \Phi_B \) generates \( E \).

\[ \oint_{\text{loop}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oint_{\text{surf}} \vec{B} \cdot d\vec{A} \]

\[ \mathcal{E} = -\frac{d\Phi_B}{dt} \]

- Generalized Ampere’s Law; \( \vec{B} \) is produced by both currents and time-dependent \( \Phi_E \).

\[ \oint_{\text{loop}} \vec{B} \cdot d\vec{l} = \mu_0 \int_{\text{surf}} \left( \vec{j} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A} \]

- The force on a (moving) charge.

\[ \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \]
Two identical concentric loops are arranged as shown in the Figure. One loop has a steady current flowing through it (provided by a power supply). When the power is turned off, in what direction does the induced current flow in loop 2?

A. It flows clockwise.
B. It flows counterclockwise.
C. There is no induced current.
D. Something else happens.
E. The answer depends on what direction the current was flowing in loop 1.
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A. They will attract each other.
B. They will repel each other.
C. There is no induced current.
D. They will neither repel nor attract each other.
E. The answer depends on the direction of current in loop 1.
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Induced E.M.F. and its consequences

1. \( \text{external } \vec{B}(t) \)

2. \( I(t) \)

3. \( \vec{B}(t) \)

\[ \text{Faraday: } \mathcal{E} = -\frac{d\Phi_B}{dt} \]
(Self) Inductance

The flux depends on the current, so a wire loop with a changing current induces an “additional” e.m.f. in itself that opposes the changes in the current and, thus, in the magnetic flux (sometimes called the back e.m.f.).

The back e.m.f.:

\[ E = -\frac{d\Phi_B}{dt} = -L \frac{dI}{dt} \]

\( L \) – the coefficient of proportionality between \( \Phi \) and \( I \), purely geometrical quantity.

**Inductance**: (or self-inductance)

\[ L \equiv \frac{\Phi}{I} \]

Units: T\(\cdot\)m\(^2\)/A = Henry (H)

This definition applies when there is a single current path (see below for an inductance of a system of distributed currents \( J(r) \)).
Inductance of a Long Solenoid

Long solenoid of radius $r$ and length $l$ with the total number of turns $N$:

The magnetic flux generated by current $I$:

$$\Phi_B = B\pi r^2 N = \left( B = \frac{\mu_0}{l} \frac{N}{I} \right) = \frac{\mu_0 N^2 I}{l} \pi r^2 = \mu_0 n^2 I \frac{l}{\pi} r^2$$

$$L = \frac{\Phi_B}{I} = \frac{\mu_0 N^2}{l} \pi r^2 = \mu_0 n^2 l \pi r^2$$

($n$ – the number of turns per unit length)

Note that
- the inductance scales with the solenoid volume $l \pi r^2$;
- the inductance scales as $N^2$: $B$ is proportional to $\frac{N}{l}$ and the net flux $\propto N$.

Let’s plug some numbers: $l = 0.1 \text{m}$, $N = 100$, $r = 0.01 \text{m}$

$$L = \mu_0 n^2 l \pi r^2 = 4\pi \cdot 10^{-7} \cdot \left( \frac{100}{0.1} \right)^2 \cdot 0.1 \cdot \pi (0.01)^2 \approx 40 \mu\text{H}$$

How significant is the induced e.m.f.? For $\frac{dl}{dt} \sim 10^4 A/s$ (let’s say, $\Delta I \sim 10 \text{ mA}$ in $1 \mu\text{s}$)

$$|\mathcal{E}| = L \frac{dl}{dt} = \frac{4 \cdot 10^{-5} \text{ H} \times 1 \cdot 10^4 \text{A}}{1 \mu\text{s}} = 0.4 \text{V}$$
A current $i$ flows through an inductor $L$ in the direction from point $b$ toward point $a$. There is zero resistance in the wires of the inductor. If the current is decreasing,

A. the potential is greater at point $a$ than at point $b$.
B. the potential is less at point $a$ than at point $b$.
C. the answer depends on the magnitude of $\frac{di}{dt}$ compared to the magnitude of $i$.
D. The answer depends on the value of the inductance $L$.
E. both C. and D. are correct.
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Inductor affects the current flow only if there is a change in current \((di/dt \neq 0)\).

The sign of \(di/dt\) determines the sign of the induced e.m.f.

The higher the frequency of current variations, the larger the induced e.m.f.
Energy of Magnetic Field

To increase a current in a solenoid, we have to do some work: **we work against the back e.m.f.**

Let’s ramp up the current from 0 to the final value $I$ ($i(t)$ is the instantaneous current value):

$$|V(t)| = \left| L \frac{di(t)}{dt} \right| \quad P(t) = V(t)i(t) = L \frac{di(t)}{dt}i(t)$$

$$dU = P(t)dt$$

$P$(power) - the rate at which energy is being delivered to the inductor from external sources.

The net work to ramp up the current from 0 to $I$:

$$U = \int_i^f P(t)dt = \int_0^l Li \cdot di = \frac{1}{2}LI^2$$

This energy is stored **in the magnetic field** created by the current.

$$L = \mu_0 n^2 l \pi r^2 = \mu_0 n^2 \cdot (volume)$$

$$U_B = \frac{1}{2} \mu_0 n^2 (volume) \cdot I^2 = (B = \mu_0 nI) = \frac{B^2}{2\mu_0} \cdot (volume) = u_B \cdot (volume)$$

The energy density in a magnetic field:

$$u_B = \frac{B^2}{2\mu_0}$$

(compare with $u_E = \frac{\varepsilon_0 E^2}{2}$)

**This is a general result** (not solenoid-specific).
**MRI scanner**: the magnetic field up to 3T within a volume ~ 1 m³.

The magnetic field energy:

$$U = volume \cdot u_B = 1m^3 \frac{(3T)^2}{2 \cdot 4\pi \cdot 10^{-7} \frac{Wb}{A \cdot m}} \approx 4MJ$$

The capacity of the LHe dewar: ~2,000 liters.

The heat of vaporization of liquid helium: 3 kJ/liter.

Thus, ~ 1000 liters will be evaporated during the quench.

Magnet quench:  [http://www.youtube.com/watch?v=tKj39eWFs10&feature=related](http://www.youtube.com/watch?v=tKj39eWFs10&feature=related)
Another (More General) Definition of Inductance

Alternative definition of inductance:

\[ U_B = \frac{1}{2} LI^2 = \int_{all\ space} \frac{B^2}{2\mu_0} d\tau \]

\[ L \equiv \frac{2U_B}{I^2} \]

Advantage: it works for a **distributed current flow** (when it’s unclear which loop we need to consider for the flux calculation).
Example: inductance per unit length for a coaxial cable

Uniform current density in the central conductor of radius $a$:

$$U_B = \frac{1}{2\mu_0} \int_a^b (B(r))^2 \, d\tau$$

$$B(r) = \begin{cases} \frac{\mu_0 I}{2\pi r}, & a < r < b \\ \frac{\mu_0 I r}{2\pi a^2}, & r < a \end{cases}$$

Straight Wires: Do They Have Inductance?

Using $L = 2U_B/I^2$:

$$U_B = \frac{1}{2\mu_0} \int_a^b \left( \frac{\mu_0 I}{2\pi r} \right)^2 \, d\tau = \frac{1}{2\mu_0} \int_a^b \left( \frac{\mu_0 I}{2\pi r} \right)^2 \cdot 2\pi r \, dr \approx (b \sim \ell) \approx \frac{\mu_0 I^2}{4\pi} \cdot 2\pi \cdot \ln \frac{\ell}{a}$$

Using $L = \Phi/I$:

$$\Phi \approx \ell \int_a^\ell \frac{\mu_0 I}{2\pi r} \, dr = \frac{\mu_0 I}{2\pi} \cdot \ell \cdot \ln \frac{\ell}{a}$$

$$L[H] \approx \frac{\mu_0}{2\pi} \cdot \ln \frac{\ell}{a} \approx 2 \cdot 10^{-7} \ell [m] \cdot \ln \frac{\ell}{a}$$

$\ell = 1cm$ $L \approx 10^{-9}H$
Inductance vs. Capacitance

\[ L = \frac{\Phi_B}{I} = \frac{2U_B}{I^2} \]

\[ U_B = \frac{1}{2} LI^2 = \frac{1}{2} \Phi_B^2 \]

\[ U_B = \frac{B^2}{2\mu_0} \cdot (\text{volume}) = u_B \cdot (\text{volume}) \]

\[ C = \frac{Q}{V} = \frac{2U_E}{V^2} \]

\[ U_E = \frac{1}{2} CV^2 = \frac{1}{2} \frac{Q^2}{C} \]

\[ U_E = \frac{\varepsilon_0 E^2}{2} \cdot (\text{volume}) = u_E \cdot (\text{volume}) \]
Mutual Inductance

Let’s consider two wire loops at rest. A time-dependent current in loop 1 produces a time-dependent magnetic field $B_1$. The magnetic flux is linked to loop 1 as well as loop 2. Faraday’s law: the time dependent flux of $B_1$ induces e.m.f. in both loops.

The e.m.f. in loop 2 due to the time-dependent $I_1$ in loop 1:

$$\mathcal{E}_2 = -\frac{d\Phi_{1\rightarrow 2}}{dt} \quad \Phi_{1\rightarrow 2} = M_{1\rightarrow 2}I_1$$

The flux of $B_1$ in loop 2 is proportional to the current $I_1$ in loop 1. The coefficient of proportionality $M_{1\rightarrow 2}$ (the so-called mutual inductance) is, similar to $L$, a purely geometrical quantity; its calculation requires, in general, complicated integration. Also, it’s possible to show that

$$M_{1\rightarrow 2} = M_{2\rightarrow 1} = M$$

so there is just one mutual inductance $M$.

$M$ can be either positive or negative, depending on the choices made for the senses of transversal about loops 1 and 2. Units of the (mutual) inductance – Henry (H).
**Calculation of Mutual Inductance**

Calculate $M$ for a pair of coaxial solenoids ($r_1 = r_2 = r$, $l_1 = l_2 = l$, $n_1$ and $n_2$ - the densities of turns).

\[
B_1 = \mu_0 n_1 I_1 \\
\Phi = B_1 \pi r^2 = \mu_0 n_1 I_1 \pi r^2
\]

the flux per one turn

\[
M = \frac{N_2 \Phi}{I_1} = N_2 \frac{\mu_0 n_1 \pi r^2}{I} = \frac{\mu_0 N_1 N_2 \pi r^2}{l}
\]

\[
M = \frac{\mu_0 n_2 l \pi r^2}{I} = \frac{\mu_0 N^2 \pi r^2}{l}
\]

\[
M = \sqrt{L_1 L_2}
\]

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**Experiment with two coaxial coils.**

The induced e.m.f. in the secondary coil is much greater if we use a ferromagnetic core: for a given current in the first coil the magnetic flux (and, thus the mutual inductance) will be $\sim K_m$ times greater.

\[
|\mathcal{E}_2| = \left| M^* \frac{dI_1}{dt} \right| = \left| K_m M \frac{dI_1}{dt} \right|
\]

no-core mutual inductance
Next time: Lecture 23. RL and LC circuits. §§ 30.4 - 6
Appendix I: Inductance of a Circular Wire Loop

**Example:** estimate the inductance of a circular wire loop with the radius of the loop, $b$, being much greater than the wire radius $a$.

The field dies off rapidly ($\sim 1/r$) with distance from the wire, so it won’t be a big mistake to approximate the loop as a straight wire of length $2\pi b$, and integrate the flux through a strip of the width $b$:

$$B \approx \frac{\mu_0 I}{2\pi r}$$

$$\Phi \approx 2\pi b \int_{a}^{b} \frac{\mu_0 I}{2\pi r} dr = \mu_0 b I \cdot \ln \frac{b}{a}$$

$$L = \frac{\Phi}{I} \approx \mu_0 b \cdot \ln \frac{b}{a}$$

$$L(b = 1m, a = 10^{-3}m) \approx 4\pi \cdot 10^{-7} \frac{Wb}{A \cdot m} 1m \cdot \ln 10^3 \approx 10\mu H$$

Note that we cannot assume that the wire radius is zero:

$$\Phi \sim \int_{0}^{b} \frac{\mu_0 I}{2\pi r} dr \sim \ln \frac{b}{0}$$

- diverges!

**What to do:** introduce some reasonably small non-zero radius.