Outline:

- Self-induction and self-inductance.
- Inductance of a solenoid.
- The energy of a magnetic field.
- Alternative definition of inductance.
- Mutual Inductance.
The energy of EM waves traveling in vacuum is stored in both $E$ and $B$ fields (in equal amounts).

Circuit elements that store the $E$ field energy - capacitors.

Circuit elements that store the $B$ field energy – inductors.

By combining capacitors and inductors, we can build the EM oscillators (e.g., generators of EM waves).
Maxwell’s Equations

- Gauss’s Law: charges are sources of electric field (and a non-zero net electric field flux through a closed surface); field lines begin and end on charges.
  \[ \int_{\text{surf}} \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\varepsilon_0} \]

- No magnetic monopoles; magnetic field lines form closed loops.
  \[ \int_{\text{surf}} \vec{B} \cdot d\vec{A} = 0 \]

- Faraday’s Law of electromagnetic induction; a time-dependent \( \Phi_B \) generates \( \vec{E} \).
  \[ \oint_{\text{loop}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_{\text{surf}} \vec{B} \cdot d\vec{A} \]
  \[ \mathcal{E} = -\frac{d\Phi_B}{dt} \]

- Generalized Ampere’s Law; \( \vec{B} \) is produced by both currents and time-dependent \( \Phi_E \).
  \[ \oint_{\text{loop}} \vec{B} \cdot d\vec{l} = \mu_0 \int_{\text{surf}} \left( j + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A} \]

- The force on a (moving) charge.
  \[ \vec{F} = q\left( \vec{E} + \vec{v} \times \vec{B} \right) \]
Induced E.M.F. and its consequences

1. External $\vec{B}(t)$

2. $I(t)$

3. $I(t)$

Faraday: $\mathcal{E} = -\frac{d\Phi_B}{dt}$
The flux depends on the current, so a wire loop with a changing current induces an “additional” e.m.f. in itself that opposes the changes in the current and, thus, in the magnetic flux (sometimes called the back e.m.f.).

The back e.m.f.:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -L \frac{dI}{dt}$$

$L$ – the coefficient of proportionality between $\Phi$ and $I$, purely geometrical quantity.

**Inductance:**
(or self-inductance)

$$L \equiv \frac{\Phi}{I}$$

Units: T·m²/A = Henry (H)

This definition applies when there is a single current path (see below for an inductance of a system of distributed currents $J(r)$).
Inductance of a Long Solenoid

Long solenoid of radius $r$ and length $\ell$ with the total number of turns $N$:

The magnetic flux generated by current $I$:

$$\Phi_B = B \pi r^2 N = \left( B = \mu_0 \frac{N}{\ell} I \right) = \frac{\mu_0 N^2 I}{\ell} \pi r^2 = \mu_0 n^2 I \ell \pi r^2$$

$$L = \frac{\Phi_B}{I} = \frac{\mu_0}{\ell} \pi r^2 = \mu_0 n^2 \ell \pi r^2$$

($n$ – the number of turns per unit length)

Note that

- the inductance scales with the solenoid volume $\ell \pi r^2$;
- the inductance scales as $N^2$: $B$ is proportional to $\frac{N}{\ell}$ and the net flux $\propto N$.

Let’s plug some numbers: $\ell=0.1\text{m}$, $N=100$, $r=0.01\text{m}$

$$L = \mu_0 n^2 \ell \pi r^2 = 4\pi \cdot 10^{-7} \cdot \left( \frac{100}{0.1} \right)^2 \cdot 0.1 \cdot \pi (0.01)^2 \approx 40\mu\text{H}$$

How significant is the induced e.m.f.? For $\frac{dI}{dt} \sim 10^4 \text{A/s}$ (let’s say, $\Delta I \sim 10\text{ mA}$ in $1\text{ \mu s}$)

$$|\mathcal{E}| = L \frac{dI}{dt} = \frac{4 \cdot 10^{-5} \text{H} \times 1 \cdot 10^4 \text{A}}{1 \text{ s}} = 0.4\text{V}$$
Inductance vs. Resistance

(a) Resistor with current $i$ flowing from $a$ to $b$: potential drops from $a$ to $b$.

\[ V_{ab} = iR > 0 \]

(b) Inductor with constant current $i$ flowing from $a$ to $b$: no potential difference.

\[ i \text{ constant: } \frac{di}{dt} = 0 \]

(c) Inductor with increasing current $i$ flowing from $a$ to $b$: potential drops from $a$ to $b$.

\[ i \text{ increasing: } \frac{di}{dt} > 0 \]

(d) Inductor with decreasing current $i$ flowing from $a$ to $b$: potential increases from $a$ to $b$.

\[ i \text{ decreasing: } \frac{di}{dt} < 0 \]

Inductor affects the current flow only if there is a change in current ($\frac{di}{dt} \neq 0$).

The sign of $\frac{di}{dt}$ determines the sign of the induced e.m.f.

The higher the frequency of current variations, the larger the induced e.m.f.
Energy of Magnetic Field

To increase a current in a solenoid, we have to do some work: we work against the back e.m.f. Let’s ramp up the current from 0 to the final value \( I \) (\( i(t) \) is the instantaneous current value):

\[
|V(t)| = \left| L \frac{di(t)}{dt} \right| \quad P(t) = V(t)i(t) = L \frac{di(t)}{dt}i(t)
\]

\[
dU = P(t)dt
\]

The net work to ramp up the current from 0 to \( I \):

\[
U = \int_{0}^{I} P(t)dt = \int_{0}^{I} Li \cdot di = \frac{1}{2}LI^2
\]

This energy is stored in the magnetic field created by the current.

\[
L = \mu_0 n^2 l \pi r^2 = \mu_0 n^2 \cdot \text{(volume)}
\]

\[
U_B = \frac{1}{2} \mu_0 n^2 \text{(volume)} \cdot I^2 = (B = \mu_0 nI) = \frac{B^2}{2\mu_0} \cdot \text{(volume)} = u_B \cdot \text{(volume)}
\]

The energy density in a magnetic field:

\[
u_B = \frac{B^2}{2\mu_0}
\]

(compare with \( u_E = \frac{\varepsilon_0 E^2}{2} \))

This is a general result (not solenoid-specific).
**Magnetic Resonance Imaging (MRI) scanner**: the magnetic field up to 3T within a volume ~ 1 m³.

The magnetic field energy:

\[
U = volume \cdot u_B = 1m^3 \frac{(3T)^2}{2 \cdot 4\pi \cdot 10^{-7} \frac{Wb}{A \cdot m}} \approx 4MJ
\]

The heat of vaporization of liquid helium: 3 kJ/liter. Thus, ~ 1000 liters will be evaporated during the quench.

Magnet quench: [http://www.youtube.com/watch?v=tKj39eWFs10&feature=related](http://www.youtube.com/watch?v=tKj39eWFs10&feature=related)
Another (More General) Definition of Inductance

Alternative definition of inductance:

\[ U_B = \frac{1}{2} LI^2 = \int_{all \ space} \frac{B^2}{2\mu_0} \, d\tau \quad \Rightarrow \quad L \equiv \frac{2U_B}{I^2} \]

Advantage: it works for a *distributed current flow* (when it’s unclear which loop we need to consider for the flux calculation).
Uniform current density in the central conductor of radius $a$:

$$U_B = \frac{1}{2\mu_0} \int_a^b (B(r))^2 \, d\tau$$

$$B(r) = \begin{cases} \frac{\mu_0 I}{2\pi r}, & a < r < b \\ \frac{\mu_0 I r}{2\pi a^2}, & r < a \end{cases}$$

**Straight Wires: Do They Have Inductance?**

Using $L = 2U_B/I^2$:

$$U_B = \frac{1}{2\mu_0} \int_a^b \left(\frac{\mu_0 I}{2\pi r}\right)^2 \, d\tau = \frac{1}{2\mu_0} \int_a^b \left(\frac{\mu_0 I}{2\pi r}\right)^2 \cdot 2\pi r \, dr \approx (b-a) \approx \frac{\mu_0 I^2}{4\pi} \cdot \ell \cdot \ln \frac{\ell}{a}$$

Using $L = \Phi/I$:

$$B \approx \frac{\mu_0 I}{2\pi r}$$

$$\Phi \approx \ell \int_a^\ell \frac{\mu_0 I}{2\pi r} \, dr = \frac{\mu_0 I}{2\pi} \cdot \ell \cdot \ln \frac{\ell}{a}$$

$$L[H] \approx \frac{\mu_0}{2\pi} \ell \cdot \ln \frac{\ell}{a} \approx 2 \cdot 10^{-7} \ell [m] \cdot \ln \frac{\ell}{a}$$

$$\ell = 1cm \quad L \approx 10^{-9}H$$
Inductance vs. Capacitance

\[ L = \frac{\Phi_B}{I} = \frac{2U_B}{I^2} \]

\[ U_B = \frac{1}{2} LI^2 = \frac{1}{2} \frac{\Phi_B^2}{L} \]

\[ U_B \cdot \text{(volume)} = u_B \cdot \text{(volume)} \]

\[ C = \frac{Q}{V} = \frac{2U_E}{V^2} \]

\[ U_E = \frac{1}{2} CV^2 = \frac{1}{2} \frac{Q^2}{C} \]

\[ U_E \cdot \text{(volume)} = u_E \cdot \text{(volume)} \]
Mutual Inductance

Let’s consider two wire loops at rest. A time-dependent current in loop 1 produces a time-dependent magnetic field $B_1$. The magnetic flux is linked to loop 1 as well as loop 2. Faraday’s law: the time dependent flux of $B_1$ induces e.m.f. in both loops.

The e.m.f. in loop 2 due to the time-dependent $I_1$ in loop 1:

$$\mathcal{E}_2 = -\frac{d\Phi_{1\rightarrow2}}{dt} \quad \Phi_{1\rightarrow2} = M_{1\rightarrow2}I_1$$

The flux of $B_1$ in loop 2 is proportional to the current $I_1$ in loop 1. The coefficient of proportionality $M_{1\rightarrow2}$ (the so-called mutual inductance) is, similar to $L$, a purely geometrical quantity; its calculation requires, in general, complicated integration. Also, it’s possible to show that

$$M_{1\rightarrow2} = M_{2\rightarrow1} = M$$

so there is just one mutual inductance $M$.

$M$ can be either positive or negative, depending on the choices made for the senses of transversal about loops 1 and 2. Units of the (mutual) inductance – Henry (H).

$$\mathcal{E}_2 = -M \frac{\partial I_1}{\partial t} \quad M = \frac{\Phi_{1\rightarrow2}}{I_1}$$
Calculation of Mutual Inductance

Calculate $M$ for a pair of coaxial solenoids ($r_1 = r_2 = r$, $l_1 = l_2 = l$, $n_1$ and $n_2$ - the densities of turns).

$$B_1 = \mu_0 n_1 I_1 \quad \Phi = B_1 \pi r^2 = \mu_0 n_1 I_1 \pi r^2$$

the flux per one turn

$$M = \frac{N_2 \Phi}{I_1} = \frac{N_2 \mu_0 n_1 \pi r^2}{I} = \frac{\mu_0 N_1 N_2 \pi r^2}{l}$$

$$L = \mu_0 n^2 l \pi r^2 = \frac{\mu_0 N^2 \pi r^2}{l} \quad \Rightarrow \quad M = \sqrt{L_1 L_2}$$

Experiment with two coaxial coils.

The induced e.m.f. in the secondary coil is much greater if we use a ferromagnetic core: for a given current in the first coil the magnetic flux (and, thus the mutual inductance) will be $\sim K_m$ times greater.

$$|\mathcal{E}_2| = \left| M^* \frac{dI_1}{dt} \right| = \left| K_m M \frac{dI_1}{dt} \right|$$

no-core mutual inductance
Next time: Lecture 23. RL and LC circuits.
§§ 30.4 - 6
Appendix I: Inductance of a Circular Wire Loop

**Example:** estimate the inductance of a circular wire loop with the radius of the loop, \( b \), being much greater than the wire radius \( a \).

The field dies off rapidly (\( \sim 1/r \)) with distance from the wire, so it won’t be a big mistake to approximate the loop as a straight wire of length \( 2\pi b \), and integrate the flux through a strip of the width \( b \):

\[
\Phi \approx 2\pi b \int_{a}^{b} \frac{\mu_0 I}{2\pi r} dr = \mu_0 I \cdot \ln \frac{b}{a} \\
L = \frac{\Phi}{I} \approx \mu_0 b \cdot \ln \frac{b}{a}
\]

\[
L(b = 1m, a = 10^{-3}m) \approx 4\pi \cdot 10^{-7} \frac{Wb}{A \cdot m} 1m \cdot \ln10^3 \approx 10\mu H
\]

Note that we cannot assume that the wire radius is zero:

\[
\Phi \sim \int_{0}^{b} \frac{\mu_0 I}{2\pi r} dr \sim \ln \frac{b}{0} - \text{diverges!}
\]

**What to do:** introduce some reasonably small non-zero radius.