Outline:

- Electromagnetic Waves in Free Space
- Transfer of EM Energy in Space: Poynting Formalism
- Radiation Pressure.

The final exam will be held on Wednesday, December 16 from 4:00 to 7:00 PM on the CAC campus. If you have a conflict (i.e., 2 exams at the same time or more than 2 exams in one 24-hour period) you must contact Professor Cizewski with your entire exam schedule at your earliest convenience but not later than 5:00 PM on Monday, December 7 to be eligible for a conflict final exam.

Iclicker scores will be posted soon. If you still see zero as your score, email professor Gershenson your name and Iclicker #.

Prof. Gustafsson will be away December 2 – 12, no office hours
TENTATIVE (!) grade boundaries for the course:

- Above 85 A
- 85 to 80 B+
- 80 to 70 B
- 70 to 65 C+
- 65 to 53 C
- 53 to 40 D
- Below 40 F

The boundaries will be finalized after the final exam.
Maxwell’s Equations in Vacuum

\[
\begin{align*}
\oint_{\text{surf}} \vec{E} \cdot d\vec{A} &= 0 \\
\oint_{\text{surf}} \vec{B} \cdot d\vec{A} &= 0 \\
\oint_{\text{loop}} \vec{E} \cdot d\vec{l} &= -\frac{d}{dt} \oint_{\text{surf}} \vec{B} \cdot d\vec{A} \\
\oint_{\text{loop}} \vec{B} \cdot d\vec{l} &= \varepsilon_0 \mu_0 \oint_{\text{surf}} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A}
\end{align*}
\]

Do these equations have non-trivial \((E \neq 0, B \neq 0)\) solutions in vacuum?

Yes, these solutions describe electromagnetic waves that propagate at the speed of light!

Maxwell’s greatest triumph: prediction of the existence of electromagnetic waves that could travel through empty space at the speed of light and identification of light as electromagnetic waves.
Generation of EM waves: by \textit{accelerated} charges and, thus, by AC currents.

Electromagnetic waves represent a novel (for us) type of solutions of Maxwell Equations. Properties of electromagnetic waves differ substantially from that of the stationary electric and magnetic fields. For instance, the amplitude of a spherical e.-m. wave dies out as $1/r$, in contrast to the $1/r^2$ dependence for stationary fields.

We’ll consider the simplest case of a plane monochromatic wave.

\[
\begin{align*}
\int_{\text{surf}} \vec{E} \cdot d\vec{A} &= 0 \\
\int_{\text{surf}} \vec{B} \cdot d\vec{A} &= 0
\end{align*}
\]

- ensure \textit{transverse} character of EM waves

\[
\begin{align*}
\int_{\text{loop}} \vec{E} \cdot d\vec{l} &= -\frac{d}{dt} \int_{\text{surf}} \vec{B} \cdot d\vec{A} \\
\int_{\text{loop}} \vec{B} \cdot d\vec{l} &= \varepsilon_0 \mu_0 \int_{\text{surf}} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A}
\end{align*}
\]

- provide the wave equations for both $E$ and $B$ waves, e.g. a plane $E$ wave that travels along $+x$:

\[
\frac{d^2 E(x,t)}{dx^2} - \varepsilon_0 \mu_0 \frac{d^2 E(x,t)}{dt^2} = 0
\]
Monochromatic e.-m. wave in vacuum, with frequency \( \omega \) and wave vector \( k = \frac{2\pi}{\lambda} = \frac{\omega}{c} \).  

Plane-front wave traveling with the phase speed \( c \):

\[
\vec{E}(x, t) = E_0 \cos(kx - \omega t)\hat{j}
\]

\[
\vec{B}(x, t) = B_0 \cos(kx - \omega t)\hat{k}
\]

\( \vec{k} \vec{r} - \omega t = \text{the phase} \)

\( \vec{k} \vec{r} = \text{const} - \text{the phase front (the surface of constant phase)} \).

1. The e.-m. wave in free space is a **transverse** wave.

\( \vec{E} \) and \( \vec{B} \) in the traveling e.-m. wave are perpendicular to the direction of propagation (\( \vec{k} \)).

2. In vacuum, the EM waves travel at the speed of light \( c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \).

Phase velocity:

\[
kx - \omega t = \text{const} \quad \text{and} \quad kdx - \omega dt = 0
\]

\[
\frac{d^2 E(x, t)}{dx^2} - \varepsilon_0 \mu_0 \frac{d^2 E(x, t)}{dt^2} = 0
\]

\[
\frac{d^2 f(x, t)}{dx^2} - \frac{1}{v_p^2} \frac{d^2 f(x, t)}{dt^2} = 0
\]

\[
v_p = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = c
\]

\[
c \approx 3 \cdot 10^8 \text{ m/s}
\]
3. The energy densities in the electric and magnetic fields in e.-m. waves are equal.

\[ B = \frac{E}{c} \quad u_B = \frac{1}{2\mu_0}B^2(r, t) \quad (p. 999) \quad u_E = \frac{\varepsilon_0}{2}E^2(r, t) \quad (p. 795) \]

- this doesn’t mean that \( B \) is “weaker” than \( E \), this depends on the units for \( E \) and \( B \) (“apples” vs. “oranges”). The meaningful comparison: the energy densities in the electric and magnetic fields that form the wave:

\[ u_E = \frac{\varepsilon_0}{2}E^2(r, t) = \frac{\varepsilon_0}{2}c^2B^2(r, t) = \left( c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} \right) = \frac{1}{2\mu_0}B^2(r, t) = u_B \]

Energy density in e.-m. waves:

\[ u_{EB} = \frac{1}{2}\left(\varepsilon_0E^2(r, t) + \frac{B^2(r, t)}{\mu_0}\right) = \varepsilon_0E^2(r, t) = \frac{B^2(r, t)}{\mu_0} \]

The average energy density in a monochromatic e.-m. waves:

\[ \langle \cos^2(kx - \omega t) \rangle_t = 1/2 \]

\[ u_{EB} = \frac{1}{2}\varepsilon_0E_0^2 = \frac{B_0^2}{2\mu_0} \]
**Intensity of Electromagnetic Waves**

**Intensity**: the average **power** transported by the e.-m. wave per unit area

Usually we are interested in the **time-averaged** quantities. For monochromatic waves

\[
\langle \cos^2(\omega t) \rangle = \left(\frac{1}{2} + \frac{1}{2} \cos(2\omega t)\right) = \frac{1}{2}:
\]

\[
\langle u_E \rangle = \frac{\varepsilon_0}{2} \langle E^2(r, t) \rangle = \frac{\varepsilon_0}{4} E_0^2 \quad E_0 - \text{the amplitude of } E(r, t) = E_0 \cos(\omega t)
\]

\[
\langle u_B \rangle = \frac{1}{4\mu_0} B_0^2 = \langle u_E \rangle \quad \langle u_{EB} \rangle = \frac{1}{2\mu_0} B_0^2 = \frac{\varepsilon_0}{2} E_0^2
\]

**Intensity**: \[ I = c \langle u_{EB} \rangle = \frac{1}{2} c \varepsilon_0 E_0^2 = \frac{1}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0^2 = \frac{1}{2} \sqrt{\frac{\mu_0}{\varepsilon_0}} B_0^2 \] Units: W/m²

**Example**: the intensity of sunlight hitting the Earth is \(~ 1,400 \) W/m². Find the amplitudes of the electric and magnetic fields in the e.-m. wave.

\[
I = \frac{1}{2} c \varepsilon_0 E_0^2 \quad E_0 = \sqrt{\frac{2I}{c \varepsilon_0}} = \sqrt{\frac{2800 \text{ W/m}^2}{3 \cdot 10^8 \text{ m/s} \times 9 \cdot 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2}} \approx 1,000 \frac{V}{m}
\]

\[
B_0 = \frac{E_0}{c} = \frac{10^3 V/m}{3 \cdot 10^8 \text{ m/s}} \approx 3 \cdot 10^{-6} T \quad (\sim 1\% \text{ of the Earth’s magnetic field})
\]
Example

In a fairly brightly-lit room, the intensity of light is about $100 \text{ W/m}^2$. What is the amplitude of the electric field of the light waves in the room? If the room is 5 m long by 4 m wide by 2.5 m high and the light intensity is fairly uniform throughout the room, about how much energy is stored in the room in the form of light waves?

\[ I = \frac{1}{2} c \varepsilon_0 E_0^2 \quad E_0 = \sqrt{\frac{2I}{c \varepsilon_0}} = \sqrt{\frac{200 \text{ W/m}^2}{3 \cdot 10^8 \text{ m/s} \times 9 \cdot 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2}} \approx 272 \text{ V/m} \]

\[ \langle u_{EB} \rangle = \frac{\varepsilon_0}{2} E_0^2 = \frac{9 \cdot 10^{-12} \text{ C}^2/\text{N} \cdot \text{m}^2}{2} \left(272 \text{ V/m} \right)^2 \approx 3.3 \cdot 10^{-7} \text{ J/m}^3 \]

Total energy: \( \langle u_{EB} \rangle \cdot \text{volume} = 3.3 \cdot \frac{10^{-7} \text{ J}}{\text{m}^3} \cdot 50 \text{ m}^3 \approx 1.7 \cdot 10^{-5} \text{ J} \)
EM waves in matter: The index of refraction

From before: \( C_{\text{filled}} = \frac{\varepsilon_0 K A}{d} \), so \( \varepsilon_0 \rightarrow \varepsilon_0 K \)

And also: \( \mu_0 \rightarrow \mu_0 K_m \)

For the velocity, \( c \rightarrow v \)

Before \( c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \)

Now \( v = \frac{1}{\sqrt{K K_m \varepsilon_0 \mu_0}} \) But for most materials \( K_m \approx 1 \)

So we obtain \( v \approx \frac{1}{\sqrt{K \varepsilon_0 \mu_0}} \approx \frac{c}{\sqrt{K}} \)

That is \( \frac{c}{v} \approx \frac{1}{\sqrt{K}} = n \), the index of refraction
Poynting Vector

\[ I = c \langle \mu_0 E B \rangle = \frac{1}{2} \sqrt{\frac{\varepsilon_0}{\mu_0}} E_0^2 = (E_0 = cB_0) = \frac{1}{2\mu_0} E_0 B_0 \]

The units are right: \( E \ [V/m], B \ [\mu_0 A/m] \rightarrow \frac{1}{\mu_0}EB \ [V \cdot A/m^2 = W/m^2] \)

From our perspective, the primary function of the electromagnetic field is transportation of energy, whether in electrical circuits or in the form of radiation (solar energy is essential to life).

Question: **how does the electromagnetic energy get from one place to another?**

In order to retain the direction of the energy flow (along \( \hat{k} \)), let’s try \( \vec{E} \times \vec{B} \):

**Poynting vector:**

\[ \vec{S} = \frac{1}{\mu_0} [\vec{E} \times \vec{B}] \]

One can show (by using vector analysis) that indeed \( S \) is related to losses of electromagnetic energy in a volume:

\[ \int_{\text{surf}} \vec{S} \cdot d\vec{A} = -\frac{\partial}{\partial t} \int_{\text{volume}} \left( \frac{\varepsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right) d\tau + \int_{\text{volume}} \vec{E} \cdot \vec{j} d\tau \]

- net flux of energy out of the volume
- net loss of the e.-m. energy in the volume
- Joule heat in the volume
Poynting formalism works well in all situations:

(a) **static** (e.g., steady currents). In statics $\vec{E}$, being a potential field, is normal to the equipotential surfaces, thus $\vec{S}$ is tangential to the equipotential surfaces.

(b) **dynamic** (note that in this case $\vec{E}$ is non-conservative, and the equipotential surfaces do not make sense).

\[
\vec{S} = \frac{1}{\mu_0} \left[ \vec{E} \times \vec{B} \right]
\]

**A few consequences:**

(a) Whenever we have both $E$ and $B$ (and $E \parallel B$), the e.-m. energy flows in space! (even if we deal with stationary fields).

(b) The flow of energy is associated with the momentum of the e.m. waves (remember, the energy has “mass”).

(c) The momentum transfer (absorption or reflection of e.m. waves) implies radiation pressure.
**General statement:** whenever there is a flow of energy (it might be field energy or any other kind of energy), the energy flowing through a unit area per unit time is equal to the momentum density \( \times c^2 \).

**Massless particles (photons)**

![Diagram of photon flow with momentum density and energy flux](image)

Momentum density

\[ p_{EB} \equiv np_{ph} : \]

Energy flux:

\[ S = \frac{n(1m^2 \cdot c \cdot 1s)}{1m^2 \cdot 1s} cp_{ph} = np_{ph}c^2 \]

Energy of a photon:

\[ E_{ph} = cp_{ph} \]

Momentum of a photon:

\[ p_{ph} \]

Radiation pressure \( P_{EB} \)

\( (the \ momentum \ transferred \ to \ 1 \ m^2 \ per \ 1s) \)

\[ P_{EB} = p_{EB}(1m^2 \cdot c \cdot 1s)/1s = \frac{S}{c} \]

\[ P_{EB} = \frac{S}{c} \]

(this is for absorption, the pressure is twice as much for reflection)

Relativity (Einstein) tells us:

\[ E^2 = (M_0c^2)^2 + (pc)^2 \]
Find the pressure of sunlight at the earth’s surface, using intensity $S = 1,000 W/m^2$ and assuming that the radiation is absorbed.

$$P = \frac{S}{c} = \frac{1,000 W/m^2}{3 \times 10^8 m/s} \approx 3 \times 10^{-6} Pa$$

Radiation pressure has had a major effect on the development of the cosmos, from the birth of the universe to ongoing formation of stars and shaping of clouds of dust and gasses on a wide range of scales.
Thank You!

Next time: Back to simpler things (inductance, reactance, AC circuits ...)

\[ \oint E \cdot dA = \frac{q_{enc}}{\varepsilon_0} \]
\[ \oint B \cdot dA = 0 \]
\[ \oint E \cdot ds = -\frac{d\Phi_B}{dt} \]
\[ \oint B \cdot ds = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} + \mu_0 i_{enc} \]