Outline:

- Self-induction and self-inductance.
- Inductance of a solenoid.
- The energy of a magnetic field.
- Alternative definition of inductance.
- Mutual Inductance.
- CH2 exam average ~54 %.

- To the right are **VERY tentative** correlations between percentage scores and letter grades in Physics 227. The instructors reserve the right to adjust these boundaries (both up and down) before final grades are assigned at the end of the semester.

<table>
<thead>
<tr>
<th>Letter Grade</th>
<th>Percentage Score</th>
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<tr>
<td>A</td>
<td>$\geq 85$</td>
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<tr>
<td>B+</td>
<td>$80 \leq B+ &lt; 85$</td>
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<tr>
<td>B</td>
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<tr>
<td>C+</td>
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<tr>
<td>C</td>
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<tr>
<td>D</td>
<td>$40 \leq D &lt; 53$</td>
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<tr>
<td>F</td>
<td>$40 &lt; F$</td>
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Maxwell’s Equations

- Gauss’s Law: charges are sources of electric field (and a non-zero net electric field flux through a closed surface); field lines begin and end on charges.

\[ \oint_{\text{surf}} \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\varepsilon_0} \]

- No magnetic monopoles; magnetic field lines form closed loops.

\[ \oint_{\text{surf}} \vec{B} \cdot d\vec{A} = 0 \]

- Faraday’s Law of electromagnetic induction; a time-dependent \( \Phi_B \) generates \( \vec{E} \).

\[ \oint_{\text{loop}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oint_{\text{surf}} \vec{B} \cdot d\vec{A} \quad \mathcal{E} = -\frac{d\Phi_B}{dt} \]

- Generalized Ampere’s Law; \( \vec{B} \) is produced by both currents and time-dependent \( \Phi_E \).

\[ \oint_{\text{loop}} \vec{B} \cdot d\vec{l} = \mu_0 \int_{\text{surf}} \left( j + \varepsilon_0 \frac{\partial\vec{E}}{\partial t} \right) \cdot d\vec{A} \]

- The force on a (moving) charge.

\[ \vec{F} = q\left( \vec{E} + \vec{v} \times \vec{B} \right) \]
Induced E.M.F. and its consequences

1. 

external $\vec{B}(t)$

2. 

$I(t)$

loop 1

loop 2

3. 

$I(t)$

loop 1

Faraday: $\mathcal{E} = -\frac{d\Phi_B}{dt}$
The flux depends on the current, so a wire loop with a changing current induces an "additional" e.m.f. in itself that opposes the changes in the current and, thus, in the magnetic flux (sometimes called the back e.m.f.).

\[ \mathcal{E} = -\frac{d\Phi_B}{dt} = -L \frac{dI}{dt} \]

\( L \) – the coefficient of proportionality between \( \Phi \) and \( I \), purely geometrical quantity.

\textbf{Inductance:} (or self-inductance) \[ L \equiv \frac{\Phi}{I} \]

Units: T·m\(^2\)/A = Henry (H)

The self-inductance \( L \) is always positive.

This definition applies when there is a single current path (see below for an inductance of a system of distributed currents \( J(r) \)).
The magnetic flux generated by current $I$:

$$\Phi_B = B\pi r^2 N = \left( B = \frac{\mu_0 N}{\ell} I \right) = \frac{\mu_0 N^2 I}{\ell} \pi r^2 = \mu_0 n^2 I \ell \pi r^2$$

Inductance $L$:

$$L = \frac{\Phi_B}{I} = \frac{\mu_0 N^2}{\ell} \pi r^2 = \mu_0 n^2 \ell \pi r^2$$

($n$ – the number of turns per unit length)

Note that:
- the inductance scales with the solenoid volume $\ell \pi r^2$;
- the inductance scales as $N^2$: $B$ is proportional to $\frac{N}{\ell}$ and the net flux $\propto N$.

Let’s plug some numbers: $\ell=0.1\text{ m}$, $N=100$, $r=0.01\text{ m}$

$$L = \mu_0 n^2 \ell \pi r^2 = 4\pi \cdot 10^{-7} \cdot \left(\frac{100}{0.1}\right)^2 \cdot 0.1 \cdot \pi (0.01)^2 \approx 40\mu\text{H}$$

How significant is the induced e.m.f.? For $\frac{dI}{dt} \sim 10^4 A/s$ (let’s say, $\Delta I \sim 1 \text{ mA}$ in $0.1 \mu\text{s}$)

$$|\mathcal{E}| = L \frac{dI}{dt} = \frac{4 \cdot 10^{-5} \text{ H} \times 1 \cdot 10^4 \text{ A}}{0.1 \mu\text{s}} = 0.4\text{ V}$$
Inductance vs. Resistance

(a) Resistor with current $i$ flowing from $a$ to $b$: potential drops from $a$ to $b$.

\[ V_{ab} = iR > 0 \]

(b) Inductor with constant current $i$ flowing from $a$ to $b$: no potential difference.

\[ V_{ab} = L \frac{di}{dt} = 0 \]

(c) Inductor with increasing current $i$ flowing from $a$ to $b$: potential drops from $a$ to $b$.

\[ V_{ab} = L \frac{di}{dt} > 0 \]

(d) Inductor with decreasing current $i$ flowing from $a$ to $b$: potential increases from $a$ to $b$.

\[ V_{ab} = L \frac{di}{dt} < 0 \]

Inductor affects the current flow only if there is a change in current ($\frac{di}{dt} \neq 0$). The sign of $\frac{di}{dt}$ determines the sign of the induced e.m.f. The higher the frequency of current variations, the larger the induced e.m.f.
Energy of Magnetic Field

To increase a current in a solenoid, we have to do some work: **we work against the back e.m.f.**

Let’s ramp up the current from 0 to the final value \( I \) \( (i(t) \) is the instantaneous current value):

\[
|V(t)| = \left| L \frac{di(t)}{dt} \right| \\
P(t) = V(t)i(t) = L \frac{di(t)}{dt} i(t)
\]

\[ dU = P(t)dt \]

\[ P \text{(power) - the rate at which energy is being delivered to the inductor from external sources.} \]

The net work to ramp up the current from 0 to \( I \):

\[ U = \int_0^I P(t)dt = \int_0^I Li \cdot di = \frac{1}{2} LI^2 \]

This energy is stored **in the magnetic field** created by the current.

\[ L = \mu_0 n^2 l \pi r^2 = \mu_0 n^2 \cdot (volume) \]

\[ U_B = \frac{1}{2} \mu_0 n^2 (volume) \cdot I^2 = (B = \mu_0 nI) = \frac{B^2}{2\mu_0} \cdot (volume) = u_B \cdot (volume) \]

The energy density in a magnetic field:

\[ u_B = \frac{B^2}{2\mu_0} \]

(compare with \( u_E = \frac{\varepsilon_0 E^2}{2} \))

This is a general result (not solenoid-specific).
Quench of a Superconducting Solenoid

**MRI scanner**: the magnetic field up to 3T within a volume \( \sim 1 \text{ m}^3 \).

The magnetic field energy:

\[
U = \text{volume} \cdot u_B = 1 \text{m}^3 \frac{(3T)^2}{2 \cdot 4\pi \cdot 10^{-7} \frac{Wb}{A \cdot m}} \approx 4 \text{MJ}
\]

The capacity of the LHe dewar: \( \sim 2,000 \) liters.

The heat of vaporization of liquid helium: 3 kJ/liter.

Thus, \( \sim 1000 \) liters will be evaporated during the quench.

Magnet quench: [http://www.youtube.com/watch?v=tKj39eWFs10&feature=related](http://www.youtube.com/watch?v=tKj39eWFs10&feature=related)
Another (More General) Definition of Inductance

Alternative definition of inductance:

\[
\frac{1}{2} LI^2 = \int_{all\ space} \frac{B^2}{2\mu_0} d\tau
\]

\[L \equiv \frac{2U_B}{I^2}\]

Advantage: it works for a \textit{distributed current flow} (when it’s unclear which loop we need to consider for the flux calculation).
Example: inductance per unit length for a coaxial cable

Uniform current density in the central conductor of radius $a$:

$$U_B = \frac{1}{2\mu_0} \int_a^b \left( B(r) \right)^2 d\tau$$

$$B(r) = \begin{cases} \frac{\mu_0 I}{2\pi r}, & a < r < b \\ \frac{\mu_0 I r}{2\pi a^2}, & r < a \end{cases}$$

Straight Wires: Do They Have Inductance?

Using $L = 2U_B/I^2$:

$$U_B = \frac{1}{2\mu_0} \int_a^b \left( \frac{\mu_0 I}{2\pi r} \right)^2 d\tau = \frac{1}{2\mu_0} \int_a^b \left( \frac{\mu_0 I}{2\pi r} \right)^2 \cdot \ell \cdot 2\pi r dr \approx (b \sim \ell) \approx \frac{\mu_0 I^2}{4\pi} \cdot \ell \cdot \ln \frac{\ell}{a}$$

Using $L = \Phi/I$:

$$B \approx \frac{\mu_0 I}{2\pi r}$$

$$\Phi \approx \ell \int_a^\ell \frac{\mu_0 I}{2\pi r} dr = \frac{\mu_0 I}{2\pi} \cdot \ell \cdot \ln \frac{\ell}{a}$$

$$L[H] \approx \frac{\mu_0}{2\pi} \ell \cdot \ln \frac{\ell}{a} \approx 2 \cdot 10^{-7} \ell [m] \cdot \ln \frac{\ell}{a}$$

$$\ell = 1cm \quad L \approx 10^{-9}H$$
Inductance vs. Capacitance

\[ L = \frac{\Phi_B}{I} = \frac{2U_B}{I^2} \]

\[ U_B = \frac{1}{2}LI^2 = \frac{1}{2} \Phi_B^2 \]

\[ U_B = \frac{B^2}{2\mu_0} \cdot \text{(volume)} = u_B \cdot \text{(volume)} \]

\[ C = \frac{Q}{V} = \frac{2U_E}{V^2} \]

\[ U_E = \frac{1}{2}CV^2 = \frac{1}{2} \frac{Q^2}{C} \]

\[ U_E = \frac{\epsilon_0E^2}{2} \cdot \text{(volume)} = u_E \cdot \text{(volume)} \]
Mutual Inductance

Let’s consider two wire loops at rest. A time-dependent current in loop 1 produces a time-dependent magnetic field $B_1$. The magnetic flux is linked to loop 1 as well as loop 2. Faraday’s law: the time dependent flux of $B_1$ induces e.m.f. in both loops.

The e.m.f. in loop 2 due to the time-dependent $I_1$ in loop 1:

$$\mathcal{E}_2 = -\frac{d\Phi_{1\rightarrow2}}{dt} \quad \Phi_{1\rightarrow2} = M_{1\rightarrow2}I_1$$

The flux of $B_1$ in loop 2 is proportional to the current $I_1$ in loop 1. The coefficient of proportionality $M_{1\rightarrow2}$ (the so-called mutual inductance) is, similar to $L$, a purely geometrical quantity; its calculation requires, in general, complicated integration. Also, it’s possible to show that

$$M_{1\rightarrow2} = M_{2\rightarrow1} = M$$

so there is just one mutual inductance $M$.

$M$ can be either positive or negative, depending on the choices made for the senses of transversal about loops 1 and 2. **Units** of the (mutual) inductance – Henry (H).
Calculate $M$ for a pair of coaxial solenoids ($r_1 = r_2 = r, l_1 = l_2 = l$, $n_1$ and $n_2$ - the densities of turns).

$$B_1 = \mu_0 n_1 I_1 \quad \Phi = B_1 \pi r^2 = \mu_0 n_1 I_1 \pi r^2$$

the flux per one turn

$$M_{1\rightarrow 2} = M_{2\rightarrow 1} = M \quad M = \frac{N_2 \Phi}{I_1} = N_2 \mu_0 n_1 \pi r^2 = \frac{\mu_0 N_1 N_2 \pi r^2}{l}$$

$$L = \mu_0 n^2 l \pi r^2 = \frac{\mu_0 N^2 \pi r^2}{l} \quad M = \sqrt{L_1 L_2}$$

Experiment with two coaxial coils.

The induced e.m.f. in the secondary coil is much greater if we use a ferromagnetic core: for a given current in the first coil the magnetic flux (and, thus the mutual inductance) will be $\sim K_m$ times greater.

$$|\mathcal{E}_2| = \left| M^* \frac{dI_1}{dt} \right| = \left| K_m M \frac{dI_1}{dt} \right|$$

no-core mutual inductance
Next time: Lecture 23. RL and LC circuits.
§§ 30.4 - 6
Appendix I: Inductance of a Circular Wire Loop

**Example:** estimate the inductance of a circular wire loop with the radius of the loop, $b$, being much greater than the wire radius $a$.

The field dies off rapidly ($\sim 1/r$) with distance from the wire, so it won’t be a big mistake to approximate the loop as a straight wire of length $2\pi b$, and integrate the flux through a strip of the width $b$:

$$B \approx \frac{\mu_0 I}{2\pi r}$$

$$\Phi \approx 2\pi b \int_a^b \frac{\mu_0 I}{2\pi r} dr = \mu_0 b I \cdot \ln \frac{b}{a}$$

$$L = \frac{\Phi}{I} \approx \mu_0 b \cdot \ln \frac{b}{a}$$

$$L(b = 1m, a = 10^{-3}m) \approx 4\pi \cdot 10^{-7} \frac{Wb}{A \cdot m} 1m \cdot \ln 10^3 \approx 10\mu H$$

Note that we cannot assume that the wire radius is zero:

$$\Phi \sim \int_0^b \frac{\mu_0 I}{2\pi r} dr \sim \ln \frac{b}{0} - \text{diverges!}$$

**What to do:** introduce some reasonably small non-zero radius.