Second Midterm common hour exam will be held Thursday, Nov 16, 9:50 PM to 11:10 PM in six locations on the Busch campus. You should go to the room corresponding to the first 3 letters of your last name.

HLL 114  AAA-FAR
HLL 116  FAT-HAL
PHY-LH   HAQ-OCA
SEC 111  OLA-SMI
SEC 117  SOK-VIT
SEC 118  VYA-ZZZ

The exam will consist of 15 multiple-choice questions that will include answers based on concepts (such as i-clickers), formulae, and simple numerical calculations, covering sections 24.4 - 29.6 in the textbook. All exams are closed book, no calculators or other electronic devices allowed. For the midterm exam, you may bring with you a single "formula sheet", one and only one handwritten sheet of paper, maximum size 8.5 x 11 inches, on which you may write any formulae or diagrams or notes that might be helpful to you during the exam. The numerical values of relevant constants will be provided to you. You should bring #2 pencils.

Additional review sessions for the exam will be held on Thursday, November 16
9-10am, 2-3pm – in ARC 332
3-8pm – in ARC 328.
Lecture 19. Maxwell’s Equations

Outline:

- Ampere’s Law must be modified in dynamics!
- Maxwell’s Fix: Displacement Current.
- Maxwell’s Equations.
- Plane EM waves.

\[
\oint E \cdot dA = \frac{q_{\text{enc}}}{\varepsilon_0} \\
\oint B \cdot dA = 0 \\
\oint E \cdot ds = -\frac{d\Phi_B}{dt} \\
\oint B \cdot ds = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt} + \mu_0 i_{\text{enc}}
\]
Recall:
\[ \oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int_{\text{surface}} \vec{B} \cdot d\vec{a} \]

1. Choose orientation of the surface.

2. This determines the “positive” direction of circulation around the loop.

3. For this orientation of the surface, \( \Phi_B \) is positive and increases \( \Rightarrow -\frac{d\Phi_B}{dt} < 0 \).

4. In order to have negative value of \( \oint \vec{E} \cdot d\vec{l} \), \( \vec{E} \) must be oriented opposite to \( d\vec{l} \).

**Lenz:** An induced current has a direction such that the **magnetic field due to the induced current** “opposes” the change in the magnetic flux that induces the current.
All erroneous!
Experiment with a magnet falling inside a copper pipe

The time-dependent $\Phi_B$ induces the e.m.f. and currents in the walls. The current directions:
- above the magnet – CCW (looking from the top)
- below the magnet – CW.

The induced currents generate a **non-uniform** magnetic field, which exerts a force on the magnet.

$$\vec{F} = m \frac{dB}{dz} \hat{z}$$

$\vec{F}_1$ (due to the “top” current): $\frac{dB}{dz} > 0$, the force is directed up.

$\vec{F}_2$ (due to the “bottom” current): $\frac{dB}{dz} > 0$, the force is also directed up.
Magnetic fields DO NOT interact!!!
(as well as the electric fields !!!)

What interacts with what:
- currents with currents (via $B$ field);
- currents with external $B$ field created by other currents and magnets;
- magnets with external $B$ field created by other currents and magnets.

$B$ fields do not attract/repel each other!
What have we learned so far?

- Gauss’s Law: charges are sources of electric field (and a non-zero net electric field flux through a closed surface); field lines begin and end on charges.

\[ \oint_{\text{surf}} \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\varepsilon_0} \]

- No magnetic monopoles; magnetic field lines form closed loops.

\[ \oint_{\text{surf}} \vec{B} \cdot d\vec{A} = 0 \]

- Faraday’s Law of electromagnetic induction; a time-dependent \( \Phi_B \) generates \( E \).

\[ \oint_{\text{loop}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oint_{\text{surf}} \vec{B} \cdot d\vec{A} \]

- Ampere’s Law; \( B \) is generated by currents.

\[ \oint_{\text{loop}} \vec{B} \cdot d\vec{l} = \mu_0 I \]

- breaks the symmetry. Changing the \( B \) field with time can induce an \( E \) field, but \( E(t) \) cannot induce a \( B \) field.

- The force on a (moving) charge.

\[ \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \]
Ampere's Law: the black sheep

\[ \int_{\text{loop}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \]

- has to be modified!

Example: spherically-symmetric radial distribution of currents.

We "inject" charge into a conducting medium. The charge "leaks out" \((Q\) is time-dependent):

\[ \frac{dQ(t)}{dt} = -4\pi r^2 J(r, t) \]

What would be the \(B\) field generated by this current distribution?

Ampere's Law predicts a non-zero magnetic field. However, due to the spherical symmetry, the magnetic field must be zero everywhere!

Something is missing. This missing part is another source of \(B\) - the time-dependent \(E\).
Displacement Current and Generalized Ampere’s Law

Maxwell: two sources of $B \Rightarrow \vec{J}$ (current of moving charges)

and $\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ (“displacement” current).

- resolves all paradoxes and makes the system of equations symmetric: not only
  the time-dependent $\vec{B}$ generates $\vec{E}$, but also the time-dependent $\vec{E}$ generates $\vec{B}$!

Displacement current:

$$\vec{J}_D = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Generalized Ampere’s Law:

(the 4th Maxwell’s Eq.)

$$\oint_{\text{loop}} \vec{B} \cdot d\vec{l} = \mu_0 I_{net\ encl} = \mu_0 \int_{\text{surf}} \left( \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A}$$

$$= \mu_0 I_{encl} + \mu_0 \varepsilon_0 \int_{\text{surf}} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A} = \mu_0 I_{encl} + \varepsilon_0 \mu_0 \frac{d\Phi_E}{dt}$$
Resolving the Paradox

**Total** current density

\[
\vec{J}_{\text{total}}(r, t) = \vec{J}(r, t) + \epsilon_0 \frac{\partial \vec{E}(r, t)}{\partial t}
\]

Gauss Law:

\[
\frac{\partial \vec{E}(r, t)}{\partial t} = \frac{1}{4\pi\epsilon_0 r^2} \frac{dQ(t)}{dt} \hat{r}
\]

Continuity equation:

\[
\vec{J}(r, t) = -\frac{1}{4\pi r^2} \frac{dQ(t)}{dt} \hat{r}
\]

Thus,

\[
\vec{J}_{\text{total}}(r, t) = \vec{J}(r, t) + \epsilon_0 \frac{\partial \vec{E}(r, t)}{\partial t} = 0
\]

and \(\vec{B} = 0\)!
Another Example: Charging a Capacitor

Consider the capacitor which is being charged by a current $I(t)$.

As the charge builds up, the $E$-field within the capacitor is obviously changing.

Regardless of the shape of the surface spanning the Amperian loop (plane vs. bulging), we should get the same result for

$$\oint \mathbf{B}(r) \cdot d\mathbf{l}$$

Ampere’s law:

$$\oint_{\text{loop}} \mathbf{B}(r) \cdot d\mathbf{l} = \mu_0 I_{\text{encl}} = \begin{cases} \mu_0 I \text{ for plane surface} \\ 0 \text{ for bulging surface} \end{cases}$$

Very bad!!
Another Example: Charging a Capacitor (cont’d)

Taking Displacement Current into account (for “bulging” surface):

\[ \oint_{\text{loop}} \mathbf{B}(r) \cdot d\mathbf{l} = \mu_0 \varepsilon_0 \int_{\text{surf}} \frac{d\vec{E}}{dt} \cdot d\mathbf{A} \]

\[ \frac{d\vec{E}}{dt} = \frac{1}{d} \frac{dV}{dt} = \frac{1}{Cd} \frac{dQ}{dt} = \left( C = \frac{\varepsilon_0 A}{d} \right) = \frac{I}{\varepsilon_0 A} \]

\[ B(r)2\pi r = \mu_0 \varepsilon_0 \int_{\text{surf}} \frac{d\vec{E}}{dt} \cdot d\mathbf{A} = \mu_0 \varepsilon_0 \frac{1}{\varepsilon_0 A} IA = \mu_0 I \]

- the same as for the plane surface intersecting the current-carrying wire
Maxwell’s Equations

- **Gauss’s Law**: charges are sources of electric field (and a non-zero net electric field flux through a closed surface); field lines begin and end on charges.
  \[ \oint_{\text{surf}} \vec{E} \cdot d\vec{A} = \frac{Q_{\text{encl}}}{\varepsilon_0} \]

- **Faraday’s Law of electromagnetic induction**: a time-dependent \( \Phi_B \) generates \( E \).
  \[ \oint_{\text{loop}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oint_{\text{surf}} \vec{B} \cdot d\vec{A} \]

- **Generalized Ampere’s Law**: \( B \) is produced by both currents and time-dependent \( \Phi_E \).
  \[ \oint_{\text{loop}} \vec{B} \cdot d\vec{l} = \mu_0 \oint_{\text{surf}} \left( j + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \cdot d\vec{A} \]

- **The force on a (moving) charge**.
  \[ \vec{F} = q \left( \vec{E} + \vec{v} \times \vec{B} \right) \]
Maxwell’s Equations

- **Gauss’s Law**
  \[ \int_{\text{surf}} \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{encl}}}{\varepsilon_0} \]

- **No magnetic monopoles**
  \[ \int_{\text{surf}} \mathbf{B} \cdot d\mathbf{A} = 0 \]

- **Faraday’s Law of electromagnetic induction**
  \[ \oint_{\text{loop}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_{\text{surf}} \mathbf{B} \cdot d\mathbf{A} \]

- **Generalized Ampere’s Law**
  \[ \oint_{\text{loop}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_{\text{surf}} \left( \mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \cdot d\mathbf{A} \]

Different types of solutions, e.g.

- “Coulomb” electric fields generated by charges at rest \( \mathbf{E} \propto \frac{1}{r^2} \);

- Spherical electromagnetic waves generated by accelerated charges \( \mathbf{E} \propto \frac{1}{r} \).

\[ u_E = \frac{\varepsilon_0 E^2}{2} \]
Generation of EM waves

Charges moving with constant velocity do not generate EM waves. Charge \textit{acceleration} is a must (either the velocity magnitude or its direction is changing, or both) $\implies$ AC currents.
Receiving Electromagnetic Waves

Dipole antenna – sensitive to time-dependent electric field in the EM wave

Magnetic dipole antenna – sensitive to time-dependent magnetic field in the EM wave

Demonstration
Receiving Electromagnetic Waves

In which situation will the electromagnetic wave be more successful in inducing a current in the wire?

In which situation will the electromagnetic wave be more successful in inducing a current in the loop?
Maxwell’s Equations in Vacuum

\[ \oint_{\text{surf}} \vec{E} \cdot d\vec{A} = 0 \]

\[ \oint_{\text{surf}} \vec{B} \cdot d\vec{A} = 0 \]

\[ \oint_{\text{loop}} \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \oint_{\text{surf}} \vec{B} \cdot d\vec{A} \]

\[ \oint_{\text{loop}} \vec{B} \cdot d\vec{l} = \varepsilon_0 \mu_0 \oint_{\text{surf}} \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A} \]

Do these equations have non-trivial \((E \neq 0, B \neq 0)\) solutions in vacuum?

Yes, these solutions describe electromagnetic waves that propagate at the speed of light!

Maxwell’s greatest triumph: prediction of the existence of electromagnetic waves that could travel through empty space at the speed of light and identification of light as electromagnetic waves.
Next time: Review for Midterm II.

Please email me your questions!
Appendix I: Experiment with a magnet falling inside a copper pipe

**Estimate of the terminal velocity** $v$.

The change in the potential energy of the magnet per 1s equals the thermal power released in the Cu walls:

$$mg \frac{\Delta h}{1s} = \frac{\mathcal{E}^2}{R} \quad \Rightarrow \quad mgv = \frac{1}{R} \left(\frac{d\Phi}{dt}\right)^2$$

$$\Phi = \pi \frac{d^2}{4} B \quad \Rightarrow \quad \frac{d\Phi}{dt} = \pi \frac{d^2}{4} B \frac{v}{d} \approx dBv$$

Reasonable assumption: strong B field is concentrated within a distance $\sim d$ along the pipe.

-Resistance of a “ring” with a diameter $d$ and cross-sectional area $dt$, where $t$ is the wall thickness.

$$R = \rho \frac{\pi d}{dt} = \frac{\pi \rho}{t}$$

$$mgv = \frac{t}{\pi \rho} (dBv)^2$$

$$v = \frac{\pi \rho mg}{d^2 B^2 t} \quad \Rightarrow \quad v = \frac{3 \cdot 10^{-8} Ohm \cdot m \cdot 1N}{(0.05m)^2 (0.1T)^2 0.01m} = 0.12 \frac{m}{s}$$

$\rho = 10^{-8} Ohm \cdot m$

$mg = 1N$

$d = 0.05m$

$t = 0.01m$
Appendix II: Difficulty of Direct Observation of the Displacement Current

It is difficult because in the “old-fashioned” experiments with highly-conducting wires and slowly-varying fields the displacement current is much smaller than the “conduction” current:

\[ \tilde{J} \sim \tilde{J}_D \quad \tilde{J} \sim \varepsilon_0 \frac{\partial E}{\partial t} \quad \sigma E \sim \varepsilon_0 \frac{E}{\tau} \quad \tau \sim \frac{\varepsilon_0}{\sigma} \sim \frac{10^{-11} C^2/N \cdot m}{10^8 1/\Omega \cdot m} \sim 10^{-19} \text{s} \]

Let’s estimate how quickly the electric field should be changed in order to make the current density and displacement current density of the same order of magnitude:

For the first time the magnetic field generated by the displacement current inside a capacitor was directly measured in 1985 (!!!) and this was the subject of a paper in one of the most prestigious physics journals, Physical Review Letters 55, 59 (1985).