Outline:

- Ampere’s Law.
- Useful symmetries.
- Magnetic fields of an infinite straight wire with current, solenoids, and a current-carrying plane.
Too Many Right-Hand Rules…

Forces on charges/currents in external $B$

\[ \vec{F}_L = q(\vec{v} \times \vec{B}) \]

\[ \vec{F} = I(d\vec{l} \times \vec{B}) \]

$B$ field due to moving charges/currents

\[ \vec{B}(r) = \frac{\mu_0}{4\pi} q \frac{\vec{v} \times \hat{r}}{r^2} \]

\[ \vec{B}(r) = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \hat{r}}{r^2} \]

\[ \vec{r} = r\hat{r} \]
Electrostatics vs. Magnetostatics

Elementary source of the static $E$ field: point charge (zero-dimensional object, scalar)

- Gauss’ Law:
  \[ \oint_{\text{surface}} \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{q_{\text{encl}}}{\varepsilon_0} \]
  - valid in electrodynamics!

  \[ \oint_{\text{loop}} \vec{E}(\vec{r}) \cdot d\vec{l} = 0 \]
  - only in electrostatics, to be modified in electrodynamics.

Elementary source of the static $B$ field: current segment (one-dimensional object, vector)

- Absence of magnetic monopoles:
  \[ \oint_{\text{surface}} \vec{B}(\vec{r}) \cdot d\vec{A} = 0 \]
  - valid in electrodynamics!

  \[ \oint_{\text{loop}} \vec{B}(\vec{r}) \cdot d\vec{l} \neq 0 \]
  ⇒ we cannot associate a scalar potential with the $B$ field.
Circulation of $B$

Path for calculation of circulation ("Amperian" loop)

The $B$ field at a distance $r$ from a straight wire with current:

$$B(r) = \frac{\mu_0 I}{2\pi r}$$

$$\Rightarrow \oint \vec{B}(\vec{r}) \cdot d\vec{l} = \mu_0 I$$

The circulation of the $B$ field $\neq 0$ if the enclosed-by-the-loop current $\neq 0$.

$$\oint \vec{B}(\vec{r}) \cdot d\vec{l} = \oint_{1} \vec{B}(\vec{r}) \cdot d\vec{l} + \oint_{2} \vec{B}(\vec{r}) \cdot d\vec{l} + \oint_{3} \vec{B}(\vec{r}) \cdot d\vec{l} + \oint_{4} \vec{B}(\vec{r}) \cdot d\vec{l}$$

$$= 0 + \frac{\mu_0 I}{2\pi} \varphi + 0 - \frac{\mu_0 I}{2\pi} \varphi = 0$$

For a loop that doesn’t enclose any current, the circulation is 0.
Ampere’s Law

Circulation of the magnetic field around any closed loop: (magnetostatics)

\[ \oint \vec{B}(r) \cdot d\vec{l} = \mu_0 I_{encl} \]  

loop

\[ I_{encl} \equiv \int_{area} j(r) \cdot d\vec{a} \]  
- the flux of current density through the surface bounded by the loop.

“Discrete” enclosed currents:

\[ I_{encl} = \sum_{i} I_i \]

Mutual orientation of the loop for calculation of B circulation and the surface for calculation of \( I_{encl} \): the curled fingers are aligned along \( d\vec{l} \), the thumb points in the direction of “positive” \( d\vec{a} \). Thus, for the currents in the Figure

\[ \sum_{i} I_i = I_1 - I_2 + I_3 \]
Gauss’ Law vs. Ampere’s Law

Similar to Gauss’ Law, Ampere’s Law is very useful whenever it is possible to reduce a 3D (vector) problem to a 1D (scalar) problem. The key is the proper symmetry of a problem.

\[ \oint \vec{E}(r) \cdot d\vec{A} = \frac{q_{encl}}{\epsilon_0} \]

\[ \oint \vec{B}(r) \cdot d\vec{l} = \mu_0 I_{encl} \]

**Symmetry:** if a charge distribution is unchanged by rotations, translations, and reflections, then the \( E \) field is also unchanged by the same transformation.

**Symmetry:** if a current distribution is unchanged by rotations and translations, then the \( B \) field is also unchanged by the same transformation.

**Exception: reflections.**

**«Useful» symmetries:**

- Axial + translational (an infinite cylinder)
- Spherical
- Infinite slab (plane)

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Gauss’ Law vs. Ampere’s Law

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| - Axial + translational (an infinite cylinder) | - Axial + translational (an infinite cylinder) |
| - Infinite solenoid | - Infinite slab (plane) |

Why not Spherical symmetry?

Imagine that you “inject” a charge \( Q \) into a conducting medium. There will be a spherically-symmetric charge flow (current). But you cannot find the \( B \) field using the Ampere’s Law. Why?

Because this is NOT a time-independent situation (there is a time-dependent electric field due to the charge flow).
Mirror Reflection Symmetry

**Electric field**
- Uniformly charged circle
- \( \vec{E} \) is parallel to the mirror plane at any point on the plane.

**Magnetic field**
- \( \vec{B} \) is perpendicular to the mirror plane at any point on the plane.
- Top view

*The Mirror Rule for Magnetic Fields:*
- If we can slice a current distribution with a mirror in such a way that the distribution looks exactly the same after we insert the mirror as before, then \( \vec{B} \) at any point on the mirror’s surface will be perpendicular to that surface.
Axial + Translational Symmetry

An infinite cylinder carrying a current whose density depends (at most) on the distance \( r \) from the axis.

Symmetries of the current distribution:
- rotations around the axis;
- translations along the axis;
- reflections across any plane containing the axis.

\( B(r) \) is tangent to a circle centered at the axis and may depend only on the distance from the axis.

Amperian loop: a circle centered at the axis
\[
\oint_{\text{loop}} \vec{B}(r) \cdot d\vec{l} = \mu_0 I_{\text{encl}}
\]

Example: A circular wire of radius \( R \) with a \textit{uniform} current density \( j = I/(\pi R^2) \):

\[
\begin{align*}
r < R: & \quad B(r)2\pi r = \mu_0 I \frac{r^2}{R^2} \quad B(r) = \frac{\mu_0 I r}{2\pi R^2} \\
r > R: & \quad B(r)2\pi r = \mu_0 I \quad B(r) = \frac{\mu_0 I}{2\pi r}
\end{align*}
\]
Appendix V: Magnetic Field of a Straight Wire Segment

Find \( B \) at a distance \( r \) from a straight wire segment carrying \( I \).

Let’s choose the origin at the point where we measure \( B \) (\( \vec{r} = 0 \)):

\[
\vec{B}(\vec{r} = 0) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l} \times (-\vec{r}')}{|-\vec{r}'|^3}
\]

Express \( d\vec{l} \) and \( \vec{r}' \) in terms of \( r \) and \( \alpha \):

\[
l = r \tan \alpha \quad dl = r \frac{1}{\cos^2 \alpha} \, d\alpha \quad r' = \frac{r}{\cos \alpha}
\]

\[
|d\vec{l} \times (-\vec{r}')| = dl \cdot r' \cdot \sin(90 + \alpha) = dl \cdot r' \cdot \cos(\alpha) = \frac{rd\alpha}{\cos^2 \alpha} \cdot r = \frac{r^2 d\alpha}{\cos^2 \alpha}
\]

\[
B(r) = \frac{\mu_0}{4\pi} I \int_{\alpha_1}^{\alpha_2} \frac{r^2}{\cos^2 \alpha} \frac{\cos^3 \alpha}{r^3} \, d\alpha = \frac{\mu_0 I}{4\pi r} \int_{\alpha_1}^{\alpha_2} \cos \alpha \, d\alpha = \frac{\mu_0 I}{4\pi r} (\sin \alpha_2 - \sin \alpha_1)
\]

**Example:** Magnetic field of a square loop with current at the center of the loop (\( \alpha_2 = \frac{\pi}{4}, \alpha_1 = -\frac{\pi}{4} \)):

\[
\vec{B}(r) = \sum_i \vec{B_i} = 4 \frac{\mu_0 I}{4\pi \alpha/2} 2\sin \frac{\pi}{4} = \frac{4\mu_0 I}{\sqrt{2\pi} \alpha}
\]
Any charge distribution that can be considered as *superposition* of symmetrical charge distributions can be treated on the basis of Ampere’s Law.

\[ b < r < c \quad B(r)2\pi r = \mu_0 \left( I - I \frac{\pi(r^2-b^2)}{\pi(c^2-b^2)} \right) = \mu_0 I \frac{c^2-r^2}{c^2-b^2} \]

\[ B(r) = \frac{\mu_0 I \ c^2 - r^2 \ c^2 - b^2}{2\pi r} \]
Field of an Infinite Solenoid

**Approximation:** the solenoid’s radius is much smaller than its length, and we evaluate the field far from the solenoid’s ends.

The solenoid carries current $I$, the number of turns per unit length is $n$.

Symmetries of the current distribution:
- rotations around the axis;
- translations along the axis;
- reflections across any plane perpendicular to the axis.

$B$ at any point is directed *along* the axis; it can depend only on the distance from the axis.

Loop 1:
\[
\oint_{\text{loop}} \vec{B}(r) \cdot d\vec{l} = B_{\text{top}}L - B_{\text{bot}}L = 0 \quad \text{- field inside is uniform}
\]

Loop 2:
\[
\oint_{\text{loop}} \vec{B}(r) \cdot d\vec{l} = B_{\text{top}}L \pm B_{\text{bot}}L = \mu_0 nIL \quad \text{- field outside is also uniform}
\]

Ampere’s Law allows us to calculate only the combination $B_{\text{top}} \pm B_{\text{bot}} = \mu_0 nI$.

Experimental fact: $B_{\text{outside}} = 0$. 

\[B_{\text{inside}} = \mu_0 nI\]
\[B_{\text{outside}} = 0\]
Field of a Finite Solenoid

\[ B_{\text{end}} = \frac{1}{2} \mu_0 I n \]

- only for a “semi-infinite” solenoid
**Infinite Slab**

Symmetries of the current distribution:
- translations along the plane;
- reflections across any $xz$ plane;
- reflections across the $yz$ plane centered at the slab.

$B$ is directed along $y$ everywhere, it is zero along the $yz$ plane centered at the slab.

Amperian loop: rectangle in the $xy$ plane of length $l$ along $y$, one side is centered at the slab ($B=0$ at the center due to symmetry)

\[ Bl = \mu_0 jx l \quad \vec{B}(x) = -\mu_0 jx\hat{j} \]

For an infinitely thin slab with the linear current density $K = 2aj$

\[ K = 2aj \]
Appendix I. Magnetic Field as a Pseudovector

The magnetic field, being a cross product of two polar (or true) vectors, $\propto \vec{r} \times \vec{I}$, is a pseudovector (or an axial vector).

A pseudovector transforms like a true vector under a proper rotation, but gains an additional sign flip under an improper rotation such as a reflection (including inversion). Geometrically the pseudovector is the opposite of its mirror image (in contrast to a polar vector, which on reflection matches its mirror image). Examples of pseudovectors: angular velocity, torque, and angular momentum.

Another example: magnetic field of a current-carrying wire loop. If the position and current of the wire are reflected across the dashed line, the magnetic field it generates would be reflected and reversed.
Appendix II: Continuous Current Distribution

\[ \vec{j}(\vec{r}) \] - local current density

\[ I_{encl} = \int_{surface} \vec{j}(r) \cdot d\vec{a} \] the flux of the current vector field

\[ \oint_{loop} \vec{B}(r) \cdot d\vec{l} = \mu_0 \int_{surface} \vec{j}(r) \cdot d\vec{a} \]

There are infinitely many possible surfaces that have the loop as their border. Which of those surfaces is to be chosen? It does not matter; for all these surfaces \( \int_{surface} \vec{j}(r) \cdot d\vec{a} \) would be the same (due to the continuity equation for charge).
Appendix III: Field of a Toroidal Solenoid

Symmetries of the current distribution:
- rotations around the center in the plane of the toroid;
- reflections across any plane perpendicular to the plane of the toroid and going through its center.

the radial component of $B$ is zero; $B$ can depend only on the distance from the axis.

Amperian loops: circles centered at the toroid’s axis

Loop 1 \[ B(r)2\pi r = 0 \quad B(r) = 0 \quad r < r_1 \]

Loop 2 \[ B(r)2\pi r = \mu_0IN \quad B(r) = \frac{\mu_0IN}{2\pi r} \quad r_1 < r < r_2 \]

Loop 3 \[ B(r)2\pi r = 0 \quad B(r) = 0 \quad r > r_2 \]