Lecture 15. Magnetic Fields of Moving Charges and Currents

Outline:

- Magnetic Field of a Moving Charge.
- Magnetic Field of Currents.
- Interaction between Two Wires with Current.

Lecture 14:

- Hall Effect.
- Magnetic Force on a Wire Segment.
- Torque on a Current-Carrying Loop.
\[ \vec{F} = F \hat{x} = m \vec{a} = m \frac{d^2 x}{dt^2} \hat{x} \]
\[ \vec{a} = \frac{d \vec{v}}{dt} \]

\[ \vec{\tau} = \tau \hat{\phi} = L \vec{a}_{ang} = L \frac{d^2 \phi}{dt^2} \hat{\phi} \]

Moment of inertia: \( L = mr^2 \)

\[ \vec{a}_{ang} \equiv \frac{d \vec{\omega}}{dt} = \dot{\omega} \hat{\phi} = \frac{d^2 \phi}{dt^2} \hat{\phi} \]

\[ \vec{\omega} \equiv \omega \hat{\phi} = \frac{d \phi}{dt} \hat{\phi} \]

\[ \vec{\omega} = \frac{\vec{r} \times \vec{v}}{r^2} \]
Imagine that you place a magnet in a uniform magnetic field. Will the magnetic field exert a net force on the magnet? If so, what is the direction of the force (Hint: use the electric dipole analogy).

A. yes, in the direction of the magnetic field
B. yes, opposite to the direction of the magnetic field
C. yes, but the direction depends on the magnet’s orientation
D. no, the net force on the magnet is zero.
An elliptical wire loop (3 turns) carrying current $I$ is in the plane of the board. Magnetic field is uniform and directed from left to right. What is the torque on the current loop?

A. 1 Nm into board
B. 1 Nm out of board
C. 3 Nm up
D. 3 Nm down
E. Zero
Iclicker Question

Which of the following statements is FALSE?

a) The magnetic torque on a current-carrying coil of wire is larger when the magnetic field is perpendicular to the plane of the coil than when the magnetic field is in the plane of the coil.

b) The magnetic force on a charged particle moving along a magnetic field line is zero.

c) The magnetic force does zero work on a charged particle moving in a magnetic field.

d) A current-carrying planar loop of wire in a constant, uniform magnetic field has zero net magnetic force on it.

e) The net magnetic flux through any closed surface is zero.
Positive charge moving with a constant velocity $v \ll c$:

- $c$ – the speed of light

\[ \vec{B}(r) = \frac{\mu_0}{4\pi} q \frac{\vec{v} \times \hat{r}}{r^2} = \frac{\mu_0}{4\pi} q \frac{\vec{v} \times \vec{r}}{r^3} \]

- the charge is at the origin at this moment

- velocity directed into the plane of the page

- a glimpse of a deep sub-structure connecting $E$ and $B$

$\mu_0 = 4\pi \cdot 10^{-7} \frac{Ns^2}{C^2}$

$\frac{\mu_0}{4\pi} = 1 \cdot 10^{-7} \frac{Ns^2}{C^2}$

$c^2 = \frac{1}{\varepsilon_0 \mu_0}$
A positive point charge is moving directly toward point $P$. The magnetic field that the point charge produces at point $P$

A. points from the charge toward point $P$.

B. points from point $P$ toward the charge.

C. is perpendicular to the line from the point charge to point $P$.

D. is zero.

E. The answer depends on the speed of the point charge.

\[
\vec{B}(r) = \frac{\mu_0}{4\pi} q \frac{\vec{v} \times \hat{r}}{r^2}
\]
Two positive point charges move side by side in the same direction with the same velocity.

What is the direction of the magnetic force that the upper point charge exerts on the lower one?

A. toward the upper point charge (the force is attractive)
B. away from the upper point charge (the force is repulsive)
C. in the direction of the velocity
D. opposite to the direction of the velocity
E. none of the above

\[ \vec{B}(r) = \frac{\mu_0}{4\pi} q \left( \frac{\vec{v} \times \vec{r}}{r^3} \right) \]

\[ \vec{F} = q \vec{v} \times \vec{B} \]
Example: a charge of 10 nC is moving in a straight line at 30 m/s. What is the max magnitude of $B$ produced by the charge at a point 10 cm from its path? What is the max magnitude of $E$?

$$B_{\text{max}} = \frac{\mu_0 qv}{4\pi r^2} = 10^{-7} \cdot 10^{-8} \cdot \frac{30}{0.1^2} = 3 \cdot 10^{-12} \text{T}$$

$$E_{\text{max}} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \approx 10^{10} \cdot 10^{-8} \cdot \frac{0.1}{0.1^2} = 10^4 \text{N/C}$$

$$\frac{B_{\text{max}}}{E_{\text{max}}} = \frac{\mu_0 \varepsilon_0 v}{c^2}$$

Magnetic force can be regarded as a relativistic correction to the electrical force.
The charge of carriers in a wire segment $Adl$:

$$q_{seg} = \sum_i q_i = neAdl$$

$$I = nev_d A$$

$$q_{seg} \vec{v}_d = I \vec{d}l$$

$$\vec{B}(r) = \frac{\mu_0}{4\pi} \frac{I \vec{d}l \times \hat{r}}{r^2}$$

- proportional to $1/r^2$, as the electric field of a point charge

Superposition:

$$\vec{B}(r) = \frac{\mu_0}{4\pi} q_{seg} \frac{\vec{v}_d \times \hat{r}}{\vec{r}^2}$$
Find the magnetic field at the center of a circular loop with current.

Contribution of a small portion of the loop:

\[ \delta \vec{B}(r) = \frac{\mu_0 I}{4\pi} \frac{dl \times \hat{r}}{R^2} \]

All contribution – in the same direction. Total \( B \):

\[ \sum \delta B_i = \frac{\mu_0 I}{4\pi} \frac{\sum dl}{R^2} \quad \sum dl = 2\pi R \]

The field at the center of the loop:

\[ B(0) = \frac{\mu_0 I}{2R} \]

For \( N \) turns with current, the field is \( N \) times stronger.
In the figure, an irregular loop of wire carrying a current lies in the plane of the paper. Suppose that the loop is distorted into some other shape while remaining in the same plane. Point P is still within the loop. Which of the following is a TRUE statement concerning this situation?

a) The magnetic field at point P will always lie in the plane of the paper.

b) It is possible that the magnetic field at point P is zero.

c) The magnetic field at point P will not change in magnitude when the loop is distorted.

d) The magnetic field at point P will not change in direction when the loop is distorted.

e) None of the other statements are true.
Iclicker Question

A wire consists of two straight sections with a semicircular section between them. If current flows in the wire as shown, what is the direction of the magnetic field at $P$ due to the current?

A. to the right  
B. to the left  
C. out of the plane of the figure  
D. into the plane of the figure  
E. misleading question — the magnetic field at $P$ is zero
Find the magnetic field at a distance \( r \) from an infinitely long straight wire with current \( I \).

Contribution of a small portion of the wire:

\[
\delta \vec{B}(r) = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \hat{r}}{r^2}
\]

All contribution – in the same direction. Total \( B \):

\[
\sum \delta B_i \text{ (see Appendix)}
\]

\[
B(r) = \frac{\mu_0 I}{2\pi r}
\]

(the \( r \)-dependence resembles that for \( E(r) \) of an infinite charged wire)

This equation will be obtained next time using Ampere’s Law.
Consider a wire bent in the hairpin shape. The wire carries a current $I$. What is the approximate magnitude of the magnetic field at point $a$?

Superposition: the field at point $a$ is a superposition of three $\vec{B}$ fields produced by the current in the semi-circle and two straight wires.

Semi-circle: \[ B(r) = \frac{1}{2} \frac{\mu_0 I}{2R} = \frac{\mu_0 I}{4R} \] (1/2 of the field of a circular loop)

Top wire: \[ B(r) = \frac{\mu_0 I}{4\pi R} \] (1/2 of the field of an infinite wire)

Bottom wire: \[ B(r) = \frac{\mu_0 I}{4\pi R} \] (1/2 of the field of an infinite wire)

\[ B(r) = \frac{\mu_0 I}{4R} + \frac{\mu_0 I}{2\pi R} = \frac{\mu_0 I}{4R} \left( 1 + \frac{2}{\pi} \right) \]
Interaction between Two Wires with Current

The magnetic field due to $I_1$ at the position of the wire with $I_2$

$$B(r) = \frac{\mu_0 I_1}{2\pi r}$$

Force per unit length:

$$\frac{F}{L} = I_2 B = \frac{\mu_0 I_1 I_2}{2\pi r}$$

Parallel (anti-parallel) currents attract (repel) each other.

SI unit and definition for electric current: The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to $2 \times 10^{-7}$ newton per meter of length.
The long, straight wire $AB$ carries a 14-A current as shown. The rectangular loop has long edges parallel to $AB$ and carries a clockwise 5-A current. What is the magnitude and direction of the net magnetic force that the straight wire $AB$ exerts on the loop?

$$F_{tot} = L \frac{\mu_0 I_1 I_2}{2\pi} \left( \frac{1}{r_1} - \frac{1}{r_2} \right)$$

A. $\frac{\mu_0}{2\pi} 5000 \text{ N to the right}$

B. $\frac{\mu_0}{2\pi} 5000 \text{ N to the left}$

C. $\frac{\mu_0}{2\pi} 2000 \text{ N upward (toward AB)}$

D. $\frac{\mu_0}{2\pi} 2000 \text{ N downward (away from AB)}$

E. misleading question — the net magnetic force is zero
**Electrostatics vs. Magnetostatics**

**Elementary source of the static $E$ field:** point charge (zero-dimensional object, scalar)

**Gauss’ Law:**
\[ \oint_{\text{surface}} \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{q_{\text{encl}}}{\varepsilon_0} \]
- valid in electrodynamics!

\[ \oint_{\text{loop}} \vec{E}(\vec{r}) \cdot d\vec{l} = 0 \]
- valid in electrostatics only(!), will be modified in electrodynamics.

**Elementary source of the static $B$ field:** current segment (one-dimensional object, vector)

**Absence of magnetic monopoles:**
\[ \oint_{\text{surface}} \vec{B}(\vec{r}) \cdot d\vec{A} = 0 \]
- valid in electrodynamics!

**B (r) = \frac{\mu_0 I}{2\pi r} \rightarrow \oint_{\text{loop}} \vec{B}(\vec{r}) \cdot d\vec{l} = \mu_0 I \neq 0**

circulation of the field $B$ along the loop

For a loop that doesn’t enclose any current, the circulation is 0.

In magnetostatics, currents are the only source of the non-zero circulation of $B$. This will be modified in electrodynamics.
Next time: Lecture 16: Ampere’s Law.
§§ 28.6- 28.8
Appendix I: Magnetic Force as a Relativistic Correction to the Electric Force

In general, electrical repulsion + magnetic attraction.

**Charges at rest** (the proton ref. frame), only electric force (repulsion):

\[ F_E = \frac{1}{4\pi\epsilon_0} \frac{q^2}{r^2} \]

**Charges in motion** (the lab. ref. frame, primed), both \( F_E \) and \( F_B \):

\[ \vec{F}'_{\perp} = \vec{F}'_{\perp E} + \vec{F}'_{\perp B} \]

According to the special theory of relativity:

\[ F'_{\perp} = \frac{1}{\gamma} F_{\perp} \]

- \( F_{\perp}' \) - force in the lab reference frame
- \( F_{\perp} \) - force in the proton reference frame

\[ E'_{\perp} = \gamma E_{\perp} \]

\[ B'_{\perp} = \gamma B_{\perp} \]

In the lab ref. frame:

\[ F'_{\perp} = \frac{1}{\gamma} \frac{1}{4\pi\epsilon_0} \frac{q^2}{r^2} \]

\[ F'_{\perp E} = \gamma \frac{1}{4\pi\epsilon_0} \frac{q^2}{r^2} \]

\[ F'_{\perp B} = F'_{\perp} - F'_{\perp E} = \left( \gamma - \frac{1}{\gamma} \right) \frac{1}{4\pi\epsilon_0} \frac{q^2}{r^2} = \gamma \left( \frac{v}{c} \right)^2 \frac{1}{4\pi\epsilon_0} \frac{q^2}{r^2} = qv\gamma \frac{\mu_0 qv}{4\pi r^2} = qvB'_{\perp} \]

**Magnetic force could be regarded as a relativistic correction to the electrical force.**
Appendix II: Integral Form of the Magnetic Field of Currents

The charge of carriers in a wire segment \( Adl \):
\[ q_{seg} = \sum_i q_i = neAdl \]

The current:
\[ I = JA = nev_d A \quad q_{seg} \vec{v}_d = I d\vec{l} \]

- \( J \) - current density, \( I d\vec{l} \) - current segment

Superposition:
\[ \vec{B}(r) = \frac{\mu_0}{4\pi} q_{seg} \frac{\vec{v}_d \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \]

Assuming the current segment is at the origin:
\[ \vec{B}(r) = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{r}}{r^3} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \hat{r}}{r^2} \]

- proportional to \( 1/r^2 \), as an electric field of a point charge

The magnetic field of a wire loop with current:
\[ \vec{B}(r) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \]
Appendix III: Magnetic Field of a Circular Loop with Current

Find the magnetic field at the center of a circular loop with current.

\[
\mathbf{B}(r) = \frac{\mu_0}{4\pi} I \oint d\mathbf{l} \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}
\]

Let’s place the origin at the center of the loop \((r = 0)\).

\[
|\mathbf{r} - \mathbf{r}'| = R \quad dl = R \, d\alpha, \quad 0 \leq \alpha \leq 2\pi
\]

\[
B(0) = \frac{\mu_0}{4\pi} I \int \frac{dl}{R^2} = \frac{\mu_0}{4\pi} \frac{I}{R} \int_0^{2\pi} d\alpha = \frac{\mu_0 I}{4\pi R} 2\pi
\]

The field at the center of the loop:

\[
B(0) = \frac{\mu_0 I}{2R}
\]

For \(N\) turns with current, the field is \(N\) times stronger.
Appendix IV: Magnetic Field of a Circular Loop with Current

A wire loop with current has a shape of a circle of radius \( a \). Find the magnetic field at a distance \( x \) from its center along the axis of symmetry.

\[
\vec{B}(r) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l} \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}
\]

Let's place the origin at the center of the loop and introduce two angles: \( \alpha \) and \( \theta \).

\[
\vec{r} = x \cdot \hat{x}
\]

\[
|\vec{r} - \vec{r}'| = \sqrt{R^2 + x^2}
\]

\[
dl = R \, d\alpha, \quad 0 \leq \alpha \leq 2\pi
\]

\[
\cos \theta = \frac{R}{\sqrt{R^2 + x^2}}
\]

\[
B_x(x) = \frac{\mu_0}{4\pi} I \int \frac{dl}{R^2 + x^2} \cos \theta = \frac{\mu_0}{4\pi} I \frac{R}{(R^2 + x^2)^{3/2}} \int_0^{2\pi} R \, d\alpha = \frac{\mu_0}{2} I \frac{R^2}{(R^2 + x^2)^{3/2}}
\]

\( B_y \) component is zero due to the symmetry.

The field at the center of the loop (\( x=0 \)):

\[
B(0) = \frac{\mu_0 I}{2R}
\]

For \( N \) turns with current, the field is \( N \) times stronger.
Appendix V: Magnetic Field of a Straight Wire Segment

Find $B$ at a distance $r$ from a straight wire segment carrying $I$.

Let’s choose the origin at the point where we measure $B$ ($\vec{r} = 0$):

$$\vec{B}(\vec{r} = 0) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l} \times (-\vec{r}')}{|-\vec{r}'|^3}$$

Express $d\vec{l}$ and $\vec{r}'$ in terms of $r$ and $\alpha$:

$$l = r \tan \alpha \quad dl = \frac{1}{\cos^2 \alpha} \, d\alpha \quad r' = \frac{r}{\cos \alpha}$$

$$|d\vec{l} \times (-\vec{r}')| = dl \cdot r' \cdot \sin (90 + \alpha) = dl \cdot r' \cdot \cos (\alpha) = \frac{rd\alpha}{\cos^2 \alpha} \cdot r = \frac{r^2 d\alpha}{\cos^2 \alpha}$$

$$B(r) = \frac{\mu_0}{4\pi} I \int_{\alpha_1}^{\alpha_2} \frac{r^2 \cos^3 \alpha}{\cos^2 \alpha r^3} \, d\alpha = \frac{\mu_0 I}{4\pi r} \int_{\alpha_1}^{\alpha_2} \cos \alpha \, d\alpha = \frac{\mu_0 I}{4\pi r} (\sin \alpha_2 - \sin \alpha_1)$$

**Example**: Magnetic field of a square loop with current at the center of the loop ($\alpha_2 = \frac{\pi}{4}, \alpha_1 = -\frac{\pi}{4}$):

$$\vec{B}(r) = \sum_i \vec{B}_i = 4 \frac{\mu_0 I}{4\pi a/2} 2\sin \frac{\pi}{4} = \frac{4\mu_0 I}{\sqrt{2}\pi a}$$
Magnetic field of an infinitely long straight wire

\[ B(r) = \frac{\mu_0 I}{4\pi r} (\sin\alpha_2 - \sin\alpha_1) \]

Infinitely long straight wire: \( \alpha_2 = \frac{\pi}{2}, \alpha_1 = -\frac{\pi}{2} \)

\[ B(r) = \frac{\mu_0 I}{2\pi r} \]

(the \( r \)-dependence resembles that for \( E(r) \) of an infinite charged wire)
Consider a wire bent in the hairpin shape. The wire carries a current $I$. What is the approximate magnitude of the magnetic field at point $a$?

Superposition: the field at point $a$ is a superposition of three $\vec{B}$ fields produced by the current in the semi-circle and two straight wires.

Semi-circle:  
$$B(r) = \frac{1}{2} \frac{\mu_0 I}{2R} = \frac{\mu_0 I}{4R} \quad (1/2 \text{ of the field of a circular loop})$$

Top wire:  
$$B(r) = \frac{\mu_0 I}{4\pi R} \int_{\alpha_1=-\pi/2}^{\alpha_2=0} \cos\alpha \, d\alpha = \frac{\mu_0 I}{4\pi R} \left[ \sin\alpha - \sin(-\pi/2) \right] = \frac{\mu_0 I}{4\pi R} \quad (1/2 \text{ of the field of an infinite wire})$$

Bottom wire:  
$$B(r) = \frac{\mu_0 I}{4\pi R} \int_{\alpha_1=0}^{\alpha_2=\pi/2} \cos\alpha \, d\alpha = \frac{\mu_0 I}{4\pi R}$$

$$B(r) = \frac{\mu_0 I}{4R} + \frac{\mu_0 I}{2\pi R} = \frac{\mu_0 I}{4R} \left( 1 + \frac{2}{\pi} \right)$$