Lecture 12. RC circuits.

Outline:

- RC circuits: charging and discharging

Prof. Torgny Gustafsson will be away October 16 - 31 and will not be available for office hours.

Prof. Michael Gershenson will be away October 20 - November 13 and will not be available for office hours.

This won’t affect the lecture schedule.
Initial state (switch open): \( i=0 \), \( q=0 \), \( V_{ab}=0 \), \( V_{bc}=0 \)

How do we know that \( V_{bc}=0 \)?

Switch = capacitor with a very small \( C \)
Capacitor Charging (qualitative), cont’d

\[ i(0) = 0, \quad q(0) = 0, \quad V_{ab}(0) = 0, \quad V_{bc}(0) = 0 \]

**Initial state** (switch open): \( i=0, \quad q=0, \quad V_{ab}=0, \quad V_{bc}=0 \)

**At t=0, we close the switch**: \( q(0)=0, \quad V_{bc}(0)=0 \)

\[ E C \]

\[ \frac{E}{R} \quad \tau(C, R, E) ? \]

\[ I(t) \equiv \frac{dq(t)}{dt} \]
Capacitor Charging (qualitative), cont’d

(a) Capacitor initially uncharged

Switch open

\[ q(t) = \frac{\varepsilon C}{\tau(C,R,\varepsilon)} t \]

\( t = 0 \) - capacitor behaves as a “short”

\( t = \infty \) - capacitor behaves as a “break”

(b) Charging the capacitor

Switch closed

\[ I(t) \equiv dq(t)/dt \]
**Charging a Capacitor**

**Initial state** (switch open): \( i=0, \; q=0, \; V_{ab}=0, \; V_{bc}=0 \)

\( i(t), q(t) \) - instantaneous values

**At \( t=0, \) we close the switch**: \( q(0)=0, \; V_{bc}(0)=0 \)

For instantaneous values: \( \mathcal{E} - i(t)R - \frac{q(t)}{C} = 0 \) \( i(t) \equiv \frac{dq(t)}{dt} \)

\[
\frac{dq(t)}{dt} + \frac{q(t)}{RC} = \frac{\mathcal{E}}{R} \quad \frac{dq(t)}{dt} = -\frac{[q(t) - \mathcal{E}C]}{RC}
\]

\[
\frac{dq(t)}{q(t) - \mathcal{E}C} = -\frac{dt}{RC} \quad \int_{0}^{t} \frac{dq(t')}{q(t') - \mathcal{E}C} = -\frac{1}{RC} \int_{0}^{t} dt'
\]

\[
\ln \left[ \frac{q(t) - \mathcal{E}C}{-\mathcal{E}C} \right] = -\frac{t}{RC} \quad q(t) = -\mathcal{E}Ce^{-\frac{t}{\tau}} + \mathcal{E}C
\]

\( q(t) = \mathcal{E}C \left( 1 - e^{-\frac{t}{\tau}} \right) \)

\( i(t) = \frac{dq(t)}{dt} = \frac{\mathcal{E}}{R} e^{-\frac{t}{\tau}} \)

\( \tau \equiv RC \quad - \text{the time constant (units of time)} \)
Charging a Capacitor (cont’d)

\[ q(t) = \varepsilon C \left( 1 - e^{-\frac{t}{\tau}} \right) \]

\[ q_\infty = \varepsilon C \]

\[ q = \varepsilon C \left( 1 - \frac{1}{e} \right) \]

For \( R=1k\Omega \) and \( C=1mF \rightarrow \tau \equiv RC = 1s \)

\[ i(t) = \frac{dq(t)}{dt} = \frac{\varepsilon}{R} e^{-\frac{t}{\tau}} \]

\[ i_0 = \frac{\varepsilon}{R} \]

\[ i = \frac{1}{e} \frac{\varepsilon}{R} \]

as if there was no capacitor!
Discharging a Capacitor

\[ \frac{dq(t)}{dt} + \frac{q(t)}{RC} = 0 \quad q(0) = q_0 = \mathcal{E}C \]

\[ \int_{\mathcal{E}C}^{q(t)} \frac{dq(t')}{q(t')} = -\frac{1}{RC} \int_{0}^{t} dt' \]

\[ \ln \left[ \frac{q(t)}{\mathcal{E}C} \right] = -\frac{t}{RC} \]

\[ q(t) = \mathcal{E}Ce^{-\frac{t}{\tau}} \]

\[ i(t) = \frac{dq(t)}{dt} = -\frac{\mathcal{E}}{R}e^{-\frac{t}{\tau}} \]

“-” means that the current flows in the direction opposite to the charging current
Pr. 26.43: Initially the capacitors are charged to 45V. At \( t=0 \) the switch is closed. Find \( t_1 \) at which the voltage across the capacitors is reduced to 10V. What is the current at this time?

The potential difference between points \( a \) and \( b \):

\[
V(t) = V(0)e^{-\frac{t}{R\Sigma C\Sigma}}
\]

\[
t_1 = R\Sigma C\Sigma \ln \left[ \frac{V(0)}{V(t_1)} \right] = 80 \Omega \cdot 35 \mu F \cdot \ln 4.5 = 0.0042 s
\]

\[
i(t_1) = \frac{V(t_1)}{R\Sigma} = \frac{10V}{80\Omega} = 0.125 A
\]
Energy Loss in Charging a Capacitor

Energy conservation:

\[ \mathcal{E}i = Ri^2 + iV_{bc} \]

- Power (work done by the battery in 1s)
- Dissipated in \( R \)
- Stored in \( C \)

\[ i(t) = \frac{\mathcal{E}}{R} e^{-\frac{t}{\tau}} \]

Total work by the battery:

\[
\int_{0}^{\infty} \mathcal{E}i(t)dt = \int_{0}^{\infty} \frac{\mathcal{E}^2}{R} e^{-t/\tau} dt = \frac{\mathcal{E}^2}{R} (-\tau)(e^{-\infty/\tau} - e^{-0/\tau}) = \mathcal{E}^2 C = \mathcal{E} q_\infty
\]

Energy dissipated in the resistor:

\[
\int_{0}^{\infty} Ri^2(t) dt = \int_{0}^{\infty} \frac{\mathcal{E}^2}{R} e^{-2t/\tau} dt = \frac{\mathcal{E}^2}{R} \left(-\frac{\tau}{2}\right)(e^{-\infty/\tau} - e^{-0/\tau}) = \frac{\mathcal{E}^2 C}{2}
\]

Half of the work done by the battery is wasted as heat (doesn’t depend on \( R \)).

Solution: slowly (in comparison with \( \tau \)) ramp up the emf in the process of charging.
Outline:

RC circuits: charging and discharging.

Next time: Lecture 13: Magnetic Field, Magnetic Forces on Moving Charges
§§ 27.1- 27.4
Let’s consider AC current (it will be considered in greater detail later in the course).

The voltage drop build-up across the capacitor during one AC period:

\[ Q = VC \approx I \frac{1}{f} < \frac{\varepsilon}{R f} \quad V < \frac{\varepsilon}{RC f} \]

Thus, if \( RCf \gg 1 \), \( V \ll \varepsilon \) and we can ignore the presence of the “break” in the circuit. On the other hand, when \( RCf \ll 1 \), \( V \approx \varepsilon \), and the capacitor indeed “breaks” the circuit.