Self Inductance

- **Observation:** When the switch is closed the current does not jump immediately to the maximum value $I_{\text{max}} = \mathcal{E}/R$. There is an induced (back) emf.

- **This effect is called self-induction because it arises from the circuit itself.**
Self Inductance

• Self-induced emf $\mathcal{E}_L$: $\mathcal{E}_L = -L \frac{dI}{dt}$ L is the inductance of the coil.

• Inductance depends on the geometry of the coil and other physical characteristics.

• The symbol for an inductor in a circuit is

• When the current is increasing: $I(t) = I_{\text{max}}(1 - e^{-t/\tau})$

• When the current is decreasing: $I(t) = I_0 e^{-t/\tau}$

• $\tau$ is the time constant: $\tau = \frac{L}{R}$
Energy in B-Field

• The total energy stored in an inductor of inductance $L$ is:

\[
U_L = \frac{1}{2}LI^2
\]

• If we consider a solenoid ($L = \mu_0 n^2 A\ell, B = \mu_0 nI$):

\[
U_L = \frac{1}{2}\mu_0 n^2 I^2 A\ell
\]

• The energy density ($u = U_L/Volume$) is then:

\[
u_B = \frac{1}{2\mu_0}B^2
\]
A current flows through an inductor $L$ as shown. If the current is increasing,

a) The potential is greater at point $a$ than at point $b$.

b) The potential is less at point $a$ than at point $b$.

c) The potential at point $a$ is the same to that at point $b$.

d) The answer depends on the inductance $L$. 
Mutual Inductance

• When the magnetic flux through an area enclosed by a circuit changes with time due to time-varying currents in nearby circuits an emf is induced.

• Consider two closely wound coils of wire.

• There is a magnetic flux passing through coil 2 due to the current $I_1$ in coil 1: $\Phi_{12}$

• The mutual inductance $M_{21}$ of coil 2 with coil 1 is:

$$M_{21} = \frac{N_2 \Phi_{12}}{I_1}$$
Mutual Inductance

- Mutual inductance depends on the geometry of both circuits and the orientation with respect to each other.
- As the circuit separation increases the flux linking the circuits decreases so the mutual inductance decreases.
- If the current $I_1$ changes with time, then the emf induced by coil 1 in coil 2 is:
  \[ \varepsilon_2 = -M_{12} \frac{dI_1}{dt} \]
  
  Similarly, if there is a current in coil 2 that changes with time:
  \[ \varepsilon_1 = -M_{21} \frac{dI_2}{dt} \]
Mutual Inductance

- The proportionality constants are equal: \( M_{12} = M_{21} = M \), then:

\[
\begin{align*}
\mathcal{E}_1 &= -M \frac{dI_2}{dt} \\
\mathcal{E}_2 &= -M \frac{dI_1}{dt}
\end{align*}
\]

- In mutual induction, the emf induced in one coil is always proportional to the rate at which the current is changing in the other coil.

- The unit for mutual inductance \( M \) (and inductance \( L \)) is the Henry.
Consider a capacitor that has an initial charge $Q_{max}$ (at $t=0$ s the switch is closed).

When the capacitor is fully charged ($t<0$):

- The energy stored in the capacitor is: $\frac{1}{2} \frac{Q_{max}^2}{C}$
- The current is zero, so no energy is stored in the inductor.

As the capacitor begins to discharge ($t>0$):

- The energy stored in the electric field decreases.
- There is a current through the circuit, so some energy is now stored in the magnetic field of the inductor.
- Energy is transferred from the electric field of the capacitor to the magnetic field of the inductor.
LC Circuit

When the capacitor is fully discharged:

• It stores no energy.

• The current reaches a maximum, so all the energy is now stored in the inductor.

• The current will continue in the same direction and decreasing in magnitude as the capacitor charges with polarity in its plates opposite that of the initial polarity.

• The capacitor eventually becomes fully charged as the current goes to zero.

• This is followed by another discharge.
Oscillations in LC Circuit

- At $t = 0$, the charge $Q$ is maximum on the capacitor $C$.
- At $t = \frac{T}{4}$, the charge $Q$ on the capacitor $C$ is zero.
- At $t = \frac{T}{2}$, the charge $Q$ is negative on the capacitor $C$.
- At $t = T$, the charge $Q$ is maximum on the capacitor $C$.

- At $t = 0$, the current $I$ is maximum.
- At $t = \frac{T}{4}$, the current $I$ is zero.
- At $t = \frac{T}{2}$, the current $I$ is negative.
- At $t = T$, the current $I$ is zero.

- At $t = 0$, the velocity $v$ of the mass $m$ is maximum.
- At $t = \frac{T}{4}$, the velocity $v$ is zero.
- At $t = \frac{T}{2}$, the velocity $v$ is negative.
- At $t = T$, the velocity $v$ is zero.
Energy in an LC Circuit

• At any arbitrary time (after the switch is closed) when the capacitor has a charge $Q < Q_{max}$ the current is $I < I_{max}$

• At this time there is energy stored in both the capacitor and the inductor.

• The sum of the two energies must equal the total initial energy (i.e. energy of the fully charged capacitor):

$$U = U_C + U_L = \frac{1}{2} \frac{Q^2}{C} + \frac{1}{2} LI^2$$
Energy in an LC Circuit

• Because we assume zero resistance the total energy of the system remains constant.

\[ U = U_C + U_L = \frac{1}{2} Q^2 C + \frac{1}{2} LI^2 \]

\[ \frac{d}{dt} \left( \frac{1}{2} \frac{Q^2}{C} + \frac{1}{2} LI^2 \right) = 0 \]

\[ \frac{Q}{C} \frac{dQ}{dt} + LI \frac{dI}{dt} = \frac{Q}{C} + L \frac{d^2 Q}{dt^2} = 0 \]

This expression is analogous to the block-spring system.

Solving for \( Q \):

\[ Q = Q_{max} \cos(\omega t + \phi) \]

\[ \omega = \frac{1}{\sqrt{LC}} \]
Consider a circuit with a power source that provides a sinusoidal voltage (i.e. the B-field inside the inductor is constantly changing). When the switch is closed the lightbulb glows steadily. If an iron rod ($\mu_{iron} \approx 5000\mu_0$) is inserted into the interior of the solenoid, the brightness of the lightbulb

a) increases. 

b) decreases. 

c) is unaffected.
Review of SHM (Physics 1B)

• The force on a mass attached to a spring at a distance $x$ from equilibrium is:
  \[ F_s = -kx \] (Hooke’s Law)

• Then, if $F_s$ is the only force acting: $-kx = ma$ (Newton’s 2\textsuperscript{nd} Law).

• Acceleration $a = \frac{d^2x}{dt}$ so:
  \[ \frac{d^2x}{dt} = -\omega^2 x \quad \text{with} \quad \omega = \sqrt{\frac{k}{m}} \]

• Which has a solution:
  \[ x(t) = x_{max}\cos(\omega t + \phi) \]

• If $T$ is the period, then $\omega T = 2\pi$, or $T = 2\pi/\omega$

• Also $f = \frac{\omega}{2\pi} = \frac{1}{T}$