Physics 227: Lecture 4
Applications of Gauss’s Law,
Conductors in Electrostatics

• Lecture 3 review:
  • Calculate the electric field through superposition as a sum of charges or integral over a charge distribution.
  • Symmetry arguments can give you the direction of the field, in some cases.
  • Motion is calculated from the (electric) force, as you learned before.
  • Flux: $\Phi = \int E \cdot dA = \int E_\perp dA = \int E \cos \phi \ dA$.
  • Gauss’s Law: for a closed surface, $\Phi = q_{\text{enclosed}}/\varepsilon_0$.
    • The flux does not depend on the size or shape of the Gaussian surface, or the position of the charges, it depends only on the total enclosed charge.
Applying Gauss’s Law

Choose a surface that reflects the symmetry of the charge distribution.

You want a surface for which \( E_\perp \) is some constant or 0!

For \( E_\perp \neq 0 \), calculate the area of the surface \( A \).

Then \( \Phi = \int E_\perp \, dA = E_\perp \, A \).

But \( \Phi = q_{\text{enclosed}} / \varepsilon_0 \).

So... \( E_\perp = q_{\text{enclosed}} / A \varepsilon_0 \).
Gauss's Law for Point Charge $q$

(a) Gaussian surface around positive charge:
positive (outward) flux

- Use a spherical Gaussian surface centered on the point charge $q$.
- Why? The electric field at the surface will be constant and perpendicular to it.
- Gauss's Law: $\Phi = \int \mathbf{E} \cdot d\mathbf{A} = q/\varepsilon_0$
- $\int \mathbf{E} \cdot d\mathbf{A} = \mathbf{E} \int d\mathbf{A} = \mathbf{E} \cdot \mathbf{A} = 4\pi r^2 \mathbf{E}$
- $4\pi r^2 \mathbf{E} = q/\varepsilon_0$
- $\mathbf{E} = q/4\pi \varepsilon_0 r^2$
- This is what we learned from Coulomb's Law + the definition of the electric field: $\mathbf{E} = \mathbf{F}_C/q_{\text{test}} = (qq_{\text{test}}/4\pi \varepsilon_0 r^2)/q_{\text{test}} = q/4\pi \varepsilon_0 r^2$
Gauss’s Law for Uniform Sphere of Charge

- Use a Gaussian sphere co-centered with the uniform sphere of charge.
- Why? The electric field at the surface will be constant and perpendicular to it.

Gauss’s Law: \( \Phi = q/\varepsilon_0 = (4/3)\pi r^3 \rho/\varepsilon_0. \)

- \( \int E \cdot dA = E \int dA = E \cdot A = 4\pi r^2 E. \)
- \( \Rightarrow E = \rho r/3\varepsilon_0. \)
- Does this agree at the surface with \( E = q/4\pi \varepsilon_0 r_0^2? \)

- \( E = \rho r_0/3\varepsilon_0 = qr/((4\pi r_0^3/3)(3\varepsilon_0)) = qr_0/(4\pi \varepsilon_0 r_0^3) \rightarrow q/4\pi \varepsilon_0 r_0^2. \) Yes, it does.

Note: \( V_{\text{sphere}} = (4/3)\pi r_0^3 \)

\( A_{\text{sphere}} = 4\pi r_0^2 \)

Outside the sphere, \( r > r_0, \) the field is identical to that of a point charge. But inside the sphere...

Assume a uniform charge density, so \( q = (4/3)\pi r_0^3 \rho \)

Thursday, September 15, 2011
Gauss's Law for Infinite Line of Charge

For an infinitely long uniform line of charge, the field is radial and perpendicular to the line: $E(\rho, \varphi, z) \to E(\rho)$.

- Use a Gaussian cylinder centered on the line charge, $\lambda$ C/m.
- Why? The electric field at the surface will be constant and perpendicular to the "side" face, and parallel to the "end" faces.
- Gauss's Law: $\Phi = \int \mathbf{E} \cdot d\mathbf{A} = q/\varepsilon_0 = \lambda h/\varepsilon_0$.
- $\int \mathbf{E} \cdot d\mathbf{A} = E \cdot A_{\text{side}} + 2 \cdot 0 \cdot A_{\text{end}} = 2\pi rhE$.
- $\Rightarrow E = \lambda/2\pi \varepsilon_0 r$. 

Thursday, September 15, 2011
Gauss's Law Inside an Infinite Cylinder of Charge

For an infinitely long uniform line of charge, the field is radial and perpendicular to the line: \( E(\rho, \phi, z) \rightarrow E(\rho) \).

- Use a Gaussian cylinder co-centered with the cylinder of charge.
- Why? The electric field at the surface will be constant and perpendicular to the "side" face, and parallel to the "end" faces.
- Gauss's Law: \( \Phi = \int E \cdot dA = q/\varepsilon_0 = \rho \pi r^2 h/\varepsilon_0 \).
- \( \int E \cdot dA = E \cdot A_{\text{side}} + 2 \cdot 0 \cdot A_{\text{end}} = 2\pi rhE \).
- \( E = \rho r/2\varepsilon_0 \).

Note: \( \lambda = \pi r_0^2 \rho \), so we have \( E = \lambda r/2\pi \varepsilon_0 r_0^2 \) inside vs \( E = \lambda r/2\pi \varepsilon_0 r \) outside.

Thursday, September 15, 2011
Gauss’s Law for Infinite Hollow Cylinder of Charge

Outside the cylinder of charge, we get as before:
\[ E = \frac{\lambda}{2\pi \varepsilon_0 r}. \]

What happens inside the cylinder of charge?

A. \( E = 0. \)
B. \( E = \frac{\lambda}{2\pi \varepsilon_0 r}. \) (Field of line charge.)
C. \( E \) cannot be determined.
D. \( E = \frac{\rho r}{3\varepsilon_0}. \) (Field in uniform sphere.)
E. \( E = \frac{\lambda \pi r}{2\varepsilon_0}. \) (Field in uniform cylinder.)
Gauss’s Law for Infinite Hollow Cylinder of Charge

Inside a hollow infinitely long cylinder of charge, like inside a spherical shell of charge, \( E = 0 \). You draw the usual Gaussian surface enclosing no charge.

A. \( E = 0 \).

B. \( E = \frac{\lambda}{2\pi\varepsilon_0 r} \). (Field of line charge.)

C. \( E \) cannot be determined.

D. \( E = \frac{\rho r}{3\varepsilon_0} \). (Field in uniform sphere.)

E. \( E = \frac{\lambda \pi r}{2\varepsilon_0} \). (Field in uniform cylinder.)
Nasty Professor Problem

- The outer infinitely long cylinder of charge has charge density $\rho_{\text{outer}}$.
- The inner infinitely long cylinder of charge has charge density $\rho_{\text{inner}}$.
- How would you work out the field everywhere?
Note on Gaussian Surfaces

- It is standard to use Gaussian spheres and Gaussian cylinders, sometimes called "pillboxes".
- There are a number of cases where the field is uniform and parallel - \( E(x, y, z) \rightarrow E_0 \) in some direction.
- In this case you need a 3d shape that has faces parallel and perpendicular to the field.
- The textbook still uses cylinders, with the end faces perpendicular to the field, and the curved side faces parallel to the field.
- But I am going to use a nonstandard Gaussian cube, in the hope that you find it simpler.
Gauss’s Law for an Infinite Plane of Charge

For an infinite uniform plane of charge, the field is perpendicular to the plane: \( E(x,y,z) \rightarrow E(z) \).

- Use, eg, a Gaussian cube with vertical and horizontal faces of area \( A \).
- Why? \( E \) will be constant and perpendicular to the horizontal faces, and parallel to the vertical faces.

- Gauss’s Law: \( \Phi = \int E \cdot dA = q/\varepsilon_0 = \sigma A/\varepsilon_0 \)
- \( \int E \cdot dA = 4\pi A + 2E \cdot A = 2EA \)
- \( \Rightarrow E = \sigma/2\varepsilon_0 \)

Charge density: \( \sigma \ C/m^2 \)
The field between the planes must be 0.

The field outside the planes must be like that outside 1 plane of charge density $2\sigma$.

Use a Gaussian cube containing both planes.

Gauss's Law: $\Phi = \int E \cdot dA = \frac{q}{\varepsilon_0} = 2\sigma A / \varepsilon_0$

$\int E \cdot dA = 4 \cdot 0 \cdot A + 2 \cdot E \cdot A = 2EA$

$\Rightarrow E = \sigma / \varepsilon_0$

This geometry can also be solved by adding (superposing) the fields of the two individual planes.
**E field for Two Infinite Planes of Charge**

Charge density: $\pm \sigma \text{ C/m}^2$
on each plane

- What is the field above, between, and below the two planes?

A. 0, 0, 0.

B. 0, $\sigma/2\varepsilon_0$, 0.

C. 0, $\sigma/\varepsilon_0$, 0.

D. $\sigma/\varepsilon_0$, $\sigma/\varepsilon_0$, $-\sigma/\varepsilon_0$.

E. $\sigma/2\varepsilon_0$, $\sigma/\varepsilon_0$, $-\sigma/2\varepsilon_0$. 
**Electric Field for Two Infinite Planes of Charge**

Charge density: $\pm \sigma \, C/m^2$
on each plane

● What is the field above, between, and below the two planes?

A. 0, 0, 0.

B. 0, $\sigma/2\varepsilon_0$, 0.

C. 0, $\sigma/\varepsilon_0$, 0.

D. $\sigma/\varepsilon_0$, $\sigma/\varepsilon_0$, $-\sigma/\varepsilon_0$.

E. $\sigma/2\varepsilon_0$, $\sigma/\varepsilon_0$, $-\sigma/2\varepsilon_0$.

This is most easily solved adding fields. The + and - planes gives fields in the same direction when between the two, but in opposite directions outside the two. So the field between is double what a single plane would give, but the field above or below the planes is 0.
Conductors

- Within an ideal conductor, electrons are free to move around.
- If there were an E field within a conductor, the electrons will move around because there is a force on them.
- If you put an electric field on a conductor, the electrons quickly rearrange and move to a configuration where the field inside the conductor vanishes, to get to a static situation.
- So all the charge on a conductor is at the surface. The interior of a conductor is neutral, and the electric field inside a conductor is 0.
- Note: field at surface of conductor must be perpendicular to surface. A parallel field component would push electrons around.
Conductors
A Charged Conductor

- The electrons are not uniformly distributed around the surface. They are distributed so that there is no field inside the conductor. A charged sphere would have a uniform charge distribution, but not the aspherical shapes shown.
- If there is no charge in the cavity, the field is always 0 inside the cavity – for a conductor. This is not true in general.

(a) Solid conductor with charge $q_C$

(b) The same conductor with an internal cavity

(c) An isolated charge $q$ placed in the cavity

The charge $q_C$ resides entirely on the surface of the conductor. The situation is electrostatic, so $\vec{E} = 0$ within the conductor.

Because $\vec{E} = 0$ at all points within the conductor, the electric field at all points on the Gaussian surface must be zero.

For $\vec{E}$ to be zero at all points on the Gaussian surface, the surface of the cavity must have a total charge $-q$. 
Conductors

Consider a spherical conductor, with a spherical cavity in the middle.

There is a charge $+q$ at the center of the cavity.

What is the electric field in the cavity, in the conductor, and outside the conductor?

A. 0, 0, 0.
B. $1/r^2$, 0, $1/r^2$.
C. $1/r^2$, 0, 0.
D. $1/r^2$, $1/r^2$, $1/r^2$.
E. 0, 0, $1/r^2$.

Another thing to think about – what are the charge distributions on the surfaces of the conductor?
Field at Surface of a Charged Conductor

(a) Solid conductor with charge $q_C$

The charge $q_C$ resides entirely on the surface of the conductor. The situation is electrostatic, so $\vec{E} = 0$ within the conductor.

- What is the field at the surface of a charged conductor?
- Draw a small Gaussian box across the surface. The box should be small enough that the surface is flat and the charge density is constant, to a good approximation, and should be $\parallel / \perp$ to the local surface.

- Gauss's Law: $\Phi = \int \vec{E} \cdot dA = 4\pi \cdot 0 \cdot A + 0 \cdot A + E \cdot A = EA$
- $\Phi = q/\varepsilon_0 = \sigma A/\varepsilon_0$
- $\Rightarrow E = \sigma/\varepsilon_0$

This is a factor of 2 larger than for a plane of charge, because all the field is to one side.
Thank you, and
See you next Monday