Outline:

- “Charging” and “discharging” an inductor in $R-L$ circuits.
- Reactance of $L-C-R$ circuits.
A steady current flows through an inductor. If the current is doubled while the inductance remains constant, the amount of energy stored in the inductor

A. increases by a factor of $\sqrt{2}$.
B. increases by a factor of 2.
C. increases by a factor of 4.
D. increases by a factor that depends on the geometry of the inductor.
E. none of the above
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$$U = \frac{1}{2} LI^2 = \frac{B^2}{2\mu_0} \cdot \text{volume}$$
“Discharging” an Inductor

Initially the switch is closed, the coil $L$ conducts a certain current (depends on the coil’s resistance $r$), and the corresponding magnetic field energy is stored in the coil.

The switch is open at $t = 0$. As the current begins to decrease, the e.m.f. is induced in the coil that opposes the change by “pushing” the current through the coil and the resistor $R$.

\[ \mathcal{E} = -L \frac{di(t)}{dt} > 0 \]

\[ -L \frac{di(t)}{dt} - i(t)R = 0, \quad \frac{di(t)}{dt} + \frac{R}{L} i(t) = 0, \quad \frac{di(t)}{i(t)} = -\frac{R}{L} dt, \quad i(t) = I_0 \exp \left( -\frac{t}{L/R} \right) \]

$I_0 = V/r$ - the steady current through the inductor (switch S is closed).

\[ \tau = \frac{L}{R} \]

- the time constant of the circuit which consists of an inductor $L$ and resistor $R$.

Let’s check that $L/R$ has units of time:

\[ \frac{L}{R} = \frac{LI^2}{RI^2} \rightarrow \frac{J}{J/s} = S \]
$I_0$ was the current through the inductor when the switch was “ON”. It could be unrelated to the net $R$ of the circuit.

The initial rate of energy dissipation: $P \approx I_0^2 R$ (the larger $R$, the faster energy decreases) – thus, $\tau \propto 1/R$. 

Looks similar, right? BUT the current $I_0 = V_0/R$ is inversely proportional to $R$! Thus, the initial rate of energy dissipation: $P \approx V_0^2/R$ (the larger $R$, the slower energy decreases) – thus, $\tau \propto R$. 

$I_0 = V_0/R$

$\tau = RC$

“current supply”

“voltage supply”
A coil with a self-inductance of 1 H is connected in parallel to a 10-Ω light bulb and a 5-V battery. Very roughly how long would it keep the light bulb lighted if the battery were disconnected from the circuit? (Choose the closest answer).

A. 10s  
B. 2s  
C. 0.5s  
D. 0.05s  
E. 0.001s
A coil with a self-inductance of 1 H is connected in parallel to a 10-Ω light bulb and a 5-V battery. Very roughly how long would it keep the light bulb lighted if the battery were disconnected from the circuit? (Choose the closest answer).

\[ \frac{L}{2R} = 0.05s \]

\[ i(t) = i_0 \exp \left( -\frac{t}{L/R} \right) \]

\[ (i(t))^2 = (i_0)^2 \exp \left( -\frac{2t}{L/R} \right) \]

A. 10s  
B. 2s  
C. 0.5s  
D. 0.05s  
E. 0.001s
“Charging” an Inductor

As an inductor opposes any decrease in the current flowing through it, it also opposes any increase in that current.

At $t = 0$ the switch is closed $[i(t = 0) = 0]$. As the current begins to increase, the e.m.f. is induced in the coil that opposes the change.

\[
\mathcal{E} - L \frac{di(t)}{dt} - i(t)R = 0
\]

\[
\frac{di(t)}{dt} + \frac{R}{L} i(t) = \frac{\mathcal{E}}{L}, \quad i(t) = \frac{\mathcal{E}}{R} \left[ 1 - \exp \left( -\frac{t}{L/R} \right) \right]
\]

$t \gg \tau$: $i(t) = \frac{\mathcal{E}}{R}$

\[
\tau = \frac{L}{R}
\]
An inductance $L$ and a resistance $R$ are connected to a source of emf as shown. When switch $S_1$ is closed, a current begins to flow. The *final* value of the current is

A. directly proportional to $RL$.
B. directly proportional to $R/L$.
C. directly proportional to $L/R$.
D. directly proportional to $1/(RL)$.
E. independent of $L$. 
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An inductance $L$ and a resistance $R$ are connected to a source of emf as shown. When switch $S_1$ is closed, a current begins to flow. The *time* required for the current to reach one-half its final value is

A. directly proportional to $RL$.
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C. directly proportional to $L/R$.
D. directly proportional to $1/(RL)$.
E. independent of $L$. 
An inductance $L$ and a resistance $R$ are connected to a source of emf as shown. When switch $S_1$ is closed, a current begins to flow. The *time* required for the current to reach one-half its final value is

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C. directly proportional to $L/R$.

D. directly proportional to $1/(RL)$.

E. independent of $L$. 
Switch S is closed at $t = 0$.

Find current $I$ at $t = 1s$.

A. $I = 0A$
B. $I = 5A$
C. $I = 10A$
D. $I = 15A$
E. $I = 20A$
Switch S is closed at $t = 0$.

Find current $I$ at $t = 1s$.

At $t \gg RC, L/R$ we can neglect the e.m.f. generated by the inductor, as well as the current through the capacitor.

Current $I = \frac{\mathcal{E}}{R_1} = 10A$. 

\[ A. \ I = 0A \]
\[ B. \ I = 5A \]
\[ C. \ I = 10A \] **(Correct Answer)**
\[ D. \ I = 15A \]
\[ E. \ I = 20A \]
So far we have considered *transient* processes in R-C and R-L circuits: the approach to the *stationary* (time-independent) state after some perturbation (switch on / off).

Now we switch to the discussion of the **AC** \((\cos(\omega t + \varphi))\) **driven circuits**, e.g. the circuits driven by an AC voltage source.

We disregard all transient processes and instead consider the **steady-state** AC currents: currents and voltages vary with time as \(\cos(\omega t + \varphi_0)\), but their **amplitudes** are \(t\)-**independent**.
Amplitudes, \textit{rms} Values, and Average Power in AC Circuits

\[ V(t) = V_0 \cos \omega t \]
\[ I(t) = I_0 \cos(\omega t + \phi) \]
\[ P(t) = V(t)I(t) \]

Currents and voltages are NOT necessarily in phase, \( \phi \) is the phase shift between \( V \) and \( I \) (the “phase angle”).

\[ P(t) = V_0 \cos \omega t \cdot I_0 \cos(\omega t + \phi) = \frac{1}{2} V_0 I_0 [\cos(2\omega t + \phi) + \cos \phi] \]

\textbf{Average power:}

\[ P_{av} \equiv \langle P(t) \rangle = \frac{1}{2} V_0 I_0 \cos \phi \]
**rms Values**

\[ P_{av} \equiv \langle P(t) \rangle = \frac{1}{2} V_0 I_0 \cos \phi \]

**Root mean square (rms):** the square root of the average of the square of the quantity:

\[ a_{rms} = \sqrt{\langle a^2(t) \rangle} \]

\[
\begin{align*}
V_{rms} &= \frac{V_0}{\sqrt{2}} \\
I_{rms} &= \frac{I_0}{\sqrt{2}}
\end{align*}
\]

\[ P_{av} = V_{rms} \cdot I_{rms} \cos \phi \]

**Example:**

**“120V” wall outlet:** \( f = 60Hz, \ \omega = 2\pi \cdot 60 \frac{rad}{s} = 377rad/s, \ T = \frac{1}{f} \approx 17ms \)

\[ V_{rms} = 120V, \ \ \ V_0 \approx 170V \]
Problem: To find current/voltage in R-L-C circuits, we need to solve differential equations.

Solution: The use of complex numbers / phasors allows us to replace linear differential equations with algebraic ones, and reduce trigonometry to algebra 😊.

We represent voltages and currents in the R-L-C circuits as the **phase vectors (phasors)** on the 2D plane. Quantity: $A(t) = A_0 \cos \omega t$. Corresponding phasor: a vector of length $A_0$ rotating **counter-clockwise** with the angular frequency $\omega$. Instantaneous value of $A(t)$ is the projection of the phasor onto the horizontal axis.

If all the quantities oscillate with the same $\omega$, we can get rid of the term $\omega t$ by using the rotating (merry-go-around) reference frame.

We’ll consider the **steady state** of AC circuits, when all amplitudes (the phasor lengths) are $t$-independent, and the only time dependence remaining is in the **single frequency** sinusoidal oscillation of voltages and currents. The angle between different phasors represents their relative ($t$-independent) phase.
Complex Numbers, Phasors

\[ Z \equiv a + ib = Re\ Z + i\ Im\ Z \]

\[ Re\ Z = r \cos \phi \]
\[ Im\ Z = r \sin \phi \]

\[ Z = re^{i\phi} \]

\[ Z^* \equiv a - ib = Re\ Z - i\ Im\ Z \]

complex conjugate of \( Z \)

Imaginary unit:
\[ i \equiv \sqrt{-1} \]
\[ i^2 = -1 \]
\[ \frac{1}{i} = -i \]

Euler’s relationship
\[ e^{i\phi} = \cos \phi + i \ \sin \phi \]

\[ e^{i\frac{\pi}{2}} = i \]
\[ e^{-i\frac{\pi}{2}} = -i \]

The absolute value (or modulus or magnitude):
\[ |Z| \equiv \sqrt{Z \cdot Z^*} \]
\[ |Z| = \sqrt{r(\cos \phi + i \sin \phi) r(\cos \phi - i \sin \phi)} \]
\[ = \sqrt{r^2 (\cos^2 \phi + \sin^2 \phi)} = r \]

**Phasor:** refer to either \( A \ e^{i(\omega t + \phi)} \) or just \( A \ e^{i(\phi)} \). In the latter case, it is understood to be a shorthand notation, encoding the amplitude and phase of an underlying sinusoid.
The use of complex numbers / phasors allows us to replace linear differential equations with algebraic ones, and reduce trigonometry to algebra:

Addition: \((a + ib) + (c + id) = (a + c) + i(b + d)\)

Multiplication: \(Ae^{i\omega_1 t} \cdot Be^{i\omega_2 t} = AB e^{i(\omega_1 + \omega_2) t}\)

Differentiation: \(\frac{d}{dt} Ae^{i\omega t} = i\omega Ae^{i\omega t}\)
Complex Power, Active Power and Reactive Power

\[ P(t) = V_0 \cos \omega t \cdot I_0 \cos(\omega t + \phi) = \frac{1}{2} V_0 I_0 \left[ \cos(2\omega t + \phi) + \cos \phi \right] \]

\[ P_{av} = V_{rms} \cdot I_{rms} \cos \phi \]

Complex power:
\[ S = V(t)I^*(t) = V_{rms} e^{i\omega t} \cdot I_{rms} e^{-i(\omega t + \phi)} = V_{rms} I_{rms} e^{-i\phi} = V_{rms} I_{rms} \left[ \cos \phi - i \sin \phi \right] \]

\[ = P + iQ \]

Average (active, real) power

Reactive power

\[ P = Re[S] = V_{rms} I_{rms} \cos \phi \]

Apparent power:
\[ |S| = \sqrt{P^2 + Q^2} \]

\[ \cos \phi = \frac{P}{|S|} \]
Resistor

AC current through a resistor and AC voltage across the resistor are always in phase.

Power dissipated in a resistor:

\[ P_{av} = \frac{1}{2} V_0 I_0 \cos \phi = \frac{1}{2} V_0 I_0 = V_{rms} I_{rms} \]

\[ \phi = 0 \]

Active (average) power:

\[ P = Re[S] = V_{rms} I_{rms} \cos \phi = V_{rms} I_{rms} \]

\[ \phi = 0 \]

If the load is purely resistive, both the current and voltage reverse their polarity at the same time. At every instant the product of voltage and current is positive or zero, with the result that the direction of energy flow does not reverse: only the active power is transferred from the power source to the load.
Reactance \((X)\) is the opposition of a circuit element to a change of electric current due to that element's \(L\), \(C\), or \(R\).

Units: Ohms.

**Resistor:** \(V = IR = IX_R\)  
\[X_R = R\]

Reactance for a resistor is the same as its resistance, it is frequency-independent.
Capacitor

Capacitor voltage LAGS current by 90°.

\[
V(t) - \frac{Q(t)}{C} = 0
\]

\[
I(t) = \frac{dQ(t)}{dt}
\]

\[
\frac{dV(t)}{dt} = \frac{1}{C} \frac{dQ(t)}{dt} = \frac{1}{C} I(t)
\]

\[
V(t) = V_0 e^{i\omega t}
\]

\[
\frac{dV(t)}{dt} = i\omega V_0 e^{i\omega t}
\]

\[
\frac{i\omega V_0 e^{i\omega t}}{C} = \frac{I_0}{C} e^{i(\omega t + \phi)}
\]

\[
i = e^{\frac{i\pi}{2}}
\]

\[
\omega V_0 e^{i(\omega t + \frac{\pi}{2})} = \frac{I_0}{C} e^{i(\omega t + \phi)} \quad \Rightarrow \quad \phi = \pi/2
\]

For a capacitor, voltage LAGS current by 90°.
Capacitor Reactance

\[ \frac{dV(t)}{dt} = \frac{1}{C} I(t) \]

\[ i \omega V_0 e^{i \omega t} = \frac{1}{C} I_0 e^{i(\omega t + \frac{\pi}{2})} \]

\[ V_0 = \frac{I_0}{\omega C} \quad V_{rms} = \frac{I_{rms}}{\omega C} \quad V_{rms} = I_{rms} X_C \]

\[ X_C = \frac{1}{\omega C} \]

\[ V(t) = I(t) (-i) X_C \]
Active (average) power:

\[ P = Re[S] = V_{rms}I_{rms} \cos \phi = 0 \]

\[ \phi = \frac{\pi}{2} \]

If the loads are purely reactive (capacitors and/or inductors, no resistive elements), then the voltage and current are 90° out of phase. For half of each cycle, \( V \cdot I \) is positive, but on the other half of the cycle, the product is negative - on average, exactly as much energy flows toward the load as flows back. There is no net transfer of energy to the load - only reactive power flows back and forth.
In an inductor, current LAGS voltage by \(90^\circ\)
**Inductor Reactance**

**Inductor:**

\[ V(t) = L \frac{dI(t)}{dt} \quad V_0 e^{i\omega t} = L i \omega I_0 e^{i(\omega t - \frac{\pi}{2})} \]

\[ V_0 = \omega L I_0 \quad V_{rms} = \omega L I_{rms} \quad V_{rms} = I_{rms} X_L \]

\[ X_L = \omega L \quad V(t) = I(t) iX_L \]
Again, the power IS NOT dissipated in an inductor: it is stored in the magnetic field of the inductor for half a period, and returned to the power source for another half.

\[ P = Re[S] = V_{rms}I_{rms} \cos \phi = 0 \]

\[ \phi = -\frac{\pi}{2} \]
Reactance - AC \((\cos(\omega t + \varphi))\) driven circuits!

**Resistor**  
\[ V = IX_R \]  
\[ X_R = R \]

**Capacitor**  
\[ X_C = \frac{1}{\omega C} \]  
\[ V(t) = I(t) (-i)X_C \]

**Inductor**  
\[ X_L = \omega L \]  
\[ V(t) = I(t)iX_L \]

Major differences between reactance and resistance: the reactance for \(L\) and \(C\) changes with frequency, it reflects (being combined with \(\pm i\)) the phase shift between \(V\) and \(I\), and it dissipates no energy.
Resonance in the $L$-$C$ circuit (L24 in detail)

**Condition for the Resonance:** reactances for $L$ and $C$ are of the same magnitude:

$$X_C = \frac{1}{\omega C} = X_L = \omega L$$

$$\frac{1}{\omega_0 C} = \omega_0 L$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$
Next time: Lecture 24. Impedance of AC circuits, §§ 31.3 - 6
Let’s check that the energy stored in the magnetic field is 100% transformed into Joule heat after switch S is open.

The initial magnetic field energy stored in the inductor:

\[ U_B = \frac{1}{2} L (I_0)^2 = \frac{1}{2} L \left( \frac{V}{r} \right)^2 \]

\( I_0 = \frac{V}{r} \) - the steady current through the inductor (switch S is closed).

Switch S is open at \( t = 0 \). The back e.m.f.: \[ |\mathcal{E}(t)| = \left| L \frac{di(t)}{dt} \right| = (R + r)i(t) \]

Power dissipated in the resistors:

\[ P = (R + r)i^2(t) \quad i(t) = \frac{V}{r} \exp \left( -\frac{t}{L/(R + r)} \right) \quad \tau = L/(R + r) \]

Net thermal energy released in the resistors:

\[
\int_0^\infty (R + r)i^2(t) dt = (R + r) \int_0^\infty \left( \frac{V}{r} \right)^2 e^{-2t/\tau} dt
\]

\[
= (R + r) \left( \frac{V}{r} \right)^2 (-\tau/2) \left( e^{-\infty/\tau} - e^{-0/\tau} \right)
= \frac{1}{2} L \left( \frac{V}{r} \right)^2
\]
Initially the switch is closed, the coil $L$ conducts a certain current, and the corresponding magnetic field energy is stored in the coil. The switch is open at $t = 0$, and the back e.m.f. $\mathcal{E}$ is induced in the coil. Can $|\mathcal{E}(t = 0)|$ be greater than $V$?

A. never

B. always

C. only if $R+r$ is sufficiently small

D. only if $R+r$ is sufficiently large

\[\mathcal{E} = -L \frac{d i(t)}{dt}\]
\[= -L \frac{d}{dt} \left( i_0 \exp \left( -\frac{t}{L/(R+r)} \right) \right)\]
\[= Li_0 \frac{(R+r)}{L} \exp \left( -\frac{t}{L/(R+r)} \right)\]
\[= \frac{V}{r} (R+r) \exp \left( -\frac{t}{L/(R+r)} \right)\]

\[i_0 = \frac{V}{r}\]