Outline:

- Ampere’s Law.
- Useful symmetries.
- Magnetic fields of an infinite straight wire with current, solenoids, and a current-carrying plane.
Forces on charges/currents in external $B$

$$\vec{F} = q(\vec{v} \times \vec{B})$$

$$\vec{F} = I(d\vec{l} \times \vec{B})$$

$B$ field due to moving charges/currents

$$\vec{B}(r) = \frac{\mu_0}{4\pi} \frac{q \vec{v} \times \hat{r}}{r^2}$$

$$\vec{B}(r) = \frac{\mu_0}{4\pi} \frac{I d\vec{l} \times \hat{r}}{r^2}$$
Electrostatics vs. Magnetostatics

Elementary source of the static \( \vec{E} \) field: point charge (zero-dimensional object, scalar)

Elementary source of the static \( \vec{B} \) field: current segment (one-dimensional object, vector)

**Gauss’ Law:**

\[
\oint_{\text{surface}} \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{q_{\text{encl}}}{\varepsilon_0}
\]

- valid in electrodynamics!

\[
\oint_{\text{loop}} \vec{E}(\vec{r}) \cdot d\vec{l} = 0
\]

- only in electrostatics, to be modified in electrodynamics.

\[
\oint_{\text{loop}} \vec{B}(\vec{r}) \cdot d\vec{l} \neq 0
\]

- cannot be an electrostatic field

**Absence of magnetic monopoles:**

\[
\oint_{\text{surface}} \vec{B}(\vec{r}) \cdot d\vec{A} = 0
\]

- valid in electrodynamics!

In general, \( \oint_{\text{loop}} \vec{B}(\vec{r}) \cdot d\vec{l} \neq 0 \)

\( \Rightarrow \) we cannot associate a scalar potential with the \( \vec{B} \) field
Circulation of $B$

The circulation of the $B$ field $\neq 0$ if the enclosed-by-the-loop current $\neq 0$.

The $B$ field at a distance $r$ from a straight wire with current $I$:

$$B(r) = \frac{\mu_0 I}{2\pi r}$$

$$\oint \vec{B}(\vec{r}) \cdot d\vec{l} = \mu_0 I$$

For a loop that doesn’t enclose any current, the circulation is 0.
Ampere’s Law

Circulation of the magnetic field around any loop: (magnetostatics)

\[
\oint \mathbf{B}(r) \cdot d\mathbf{l} = \mu_0 I_{encl}
\]

\(I_{encl} \equiv \int \mathbf{j}(r) \cdot d\mathbf{a}\) \(-\) the flux of current density through the surface bounded by the loop.

“Discrete” enclosed currents: \(I_{encl} = \sum_i I_i\)

**Mutual orientation of the loop for calculation of \(B\) circulation and the surface for calculation of \(I_{encl}\):** the curled fingers are aligned along \(d\mathbf{l}\), the thumb points in the direction of “positive” \(d\mathbf{a}\). Thus, for the currents in the Figure

\[
\sum_i I_i = I_1 - I_2 + I_3
\]
The figure shows, in cross section, three conductors that carry currents perpendicular to the plane of the figure.

If the currents $I_1$, $I_2$, and $I_3$ all have the same magnitude, for which path(s) is the line integral of the magnetic field equal to zero?

A. path $a$ only
B. paths $a$ and $c$
C. paths $b$ and $d$
D. paths $a$, $b$, $c$, and $d$
E. The answer depends on whether the integral goes clockwise or counterclockwise around the path.
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Gauss’ Law vs. Ampere’s Law

Similar to Gauss’ Law, Ampere’s Law is very useful whenever it is possible to reduce a 3D (vector) problem to a 1D (scalar) problem. The key is the proper symmetry of a problem.

\[ \oint \vec{E}(r) \cdot d\vec{A} = \frac{q_{encl}}{\epsilon_0} \]  

\[ \oint \vec{B}(r) \cdot d\vec{l} = \mu_0 I_{encl} \]

**Symmetry:** if a charge distribution is unchanged by rotations, translations, and reflections, then the \( E \) field is also unchanged by the same transformation.

**Symmetry:** if a current distribution is unchanged by rotations and translations, then the \( B \) field is also unchanged by the same transformation.

**Exception:** reflections.

«Useful» symmetries:

- Axial + translational (an infinite cylinder)
- Spherical
- Infinite slab (plane)

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Gauss’ Law vs. Ampere’s Law

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Why not Spherical symmetry?
Imagine that you “inject” a charge $Q$ into a conducting medium. There will be a spherically-symmetric charge flow (current). But you cannot find the $B$ field using the Ampere’s Law. Why?

Because this is NOT a time-independent situation (there is a time-dependent electric field due to the charge flow).
Mirror Reflection Symmetry

Electric field

\[ \vec{E} \text{ is parallel to the mirror plane at any point on the plane.} \]

Magnetic field

\[ \vec{B} \text{ is perpendicular to the mirror plane at any point on the plane.} \]

The Mirror Rule for Magnetic Fields:

if we can slice a current distribution with a mirror in such a way that the distribution looks exactly the same after we insert the mirror as before, then \( \vec{B} \) at any point on the mirror’s surface will be perpendicular to that surface.
Axial + Translational Symmetry

An infinite cylinder carrying a current whose density depends (at most) on the distance $r$ from the axis.

Symmetries of the current distribution:
- rotations around the axis;
- translations along the axis;
- reflections across any plane containing the axis.

$B(r)$ is tangent to a circle centered at the axis and may depend only on the distance from the axis.

Amperian loop: a circle centered at the axis

$$\oint_{\text{loop}} \vec{B}(r) \cdot d\vec{l} = \mu_0 I_{\text{encl}}$$

**Example:** A circular wire of radius $R$ with a uniform current density $j = I / (\pi R^2)$:

$$r < R: \quad B(r)2\pi r = \mu_0 I \frac{r^2}{R^2} \quad B(r) = \frac{\mu_0 Ir}{2\pi R^2} \quad r > R: \quad B(r)2\pi r = \mu_0 I \quad B(r) = \frac{\mu_0 I}{2\pi r}$$
Any charge distribution that can be considered as **superposition** of symmetrical charge distributions can be treated on the basis of Ampere’s Law.

\[
b < r < c \quad B(r)2\pi r = \mu_0 \left( I - I \frac{\pi (r^2 - b^2)}{\pi (c^2 - b^2)} \right) = \mu_0 I \frac{c^2 - r^2}{c^2 - b^2}
\]

\[
B(r) = \frac{\mu_0 I}{2\pi r} \frac{c^2 - r^2}{c^2 - b^2}
\]
**Field of an Infinite Solenoid**

**Approximation**: the solenoid’s radius is much smaller than its length, and we evaluate the field far from the solenoid’s ends.

The solenoid carries current \( I \), the number of turns per unit length is \( n \).

Symmetries of the current distribution:
- rotations around the axis;
- translations along the axis;
- reflections across any plane perpendicular to the axis.

\( B \) at any point is directed **along** the axis; it can depend only on the distance from the axis.

Loop 1:
\[
\int_{\text{loop}} \vec{B}(r) \cdot d\vec{l} = B_{\text{top}}L - B_{\text{bot}}L = 0
\]
- field inside is **uniform**

Loop 2:
\[
\int_{\text{loop}} \vec{B}(r) \cdot d\vec{l} = B_{\text{top}}L \pm B_{\text{bot}}L = \mu_0 n IL
\]
- field outside is also **uniform**

Ampere’s Law allows us to calculate only the **combination** \( B_{\text{top}} \pm B_{\text{bot}} = \mu_0 n I \).

Experimental fact: \( B_{\text{outside}} = 0 \).
Field of a Finite Solenoid

\[ B_{\text{end}} = \frac{1}{2} \mu_0 I n \]

- only for a “semi-infinite” solenoid
Symmetries of the current distribution:
- translations along the plane;
- reflections across any $xz$ plane
- reflections across the $yz$ plane centered at the slab.

$B$ is directed along $y$ everywhere, it is zero along the $yz$ plane centered at the slab.

Amperian loop: rectangle in the $xy$ plane of length $l$ along $y$, one side is centered at the slab ($B=0$ at the center due to symmetry)

\[
Bl = \mu_0 j xl \\
\vec{B}(x) = -\mu_0 j x \hat{j}
\]

For an infinitely thin slab with the linear current density $K = 2aj$ (current per unit length)

\[
K = 2aj
\]
Pressure on Solenoid Walls

\[ B_{in} = \mu_0 nI = \mu_0 K \]

\[ K = nI - \text{linear current density (per unit length)} \]

Total force per unit area \(1m^2\) (pressure) on a current-carrying sheet:

\[ P = nI B_{ext} = KB_{ext} = K \frac{\mu_0 K}{2} = \frac{1}{2\mu_0} B_{in}^2 \]

For a 10T solenoid:

\[ P = \frac{10^7}{8\pi} 10^2 Pa \approx 4 \cdot 10^7 Pa \approx 400bar \]

Ultra-strong mag. fields: problem of mechanical stability of solenoids
Axial + Translational Symmetry

\[ r < R: B(r) = \frac{\mu_0 I r}{2\pi R^2} \]

\[ r > R: B(r) = \frac{\mu_0 I}{2\pi r} \]

Infinite Solenoid

\[ B_{\text{inside}} = \mu_0 n I \]
\[ B_{\text{outside}} = 0 \]

Toroidal Solenoid

\[ B(r) = \frac{\mu_0 I N}{2\pi r} \]
\[ r_1 < r < r_2 \]

Infinite Slab

Infinitely thin slab with linear current density \( K \)

\[ B = \mu_0 \frac{K}{2} \]
The magnetic field, being a cross product of two polar (or true) vectors, $\propto \vec{r} \times \vec{I}$, is a pseudovector (or an axial vector).

A pseudovector transforms like a true vector under a **proper rotation**, but gains an additional sign flip under an **improper rotation** such as a reflection (including inversion). Geometrically the pseudovector is the opposite of its mirror image (in contrast to a polar vector, which on reflection matches its mirror image). Examples of pseudovectors: **angular velocity, torque, and angular momentum**.

Another example: magnetic field of a current-carrying wire loop. If the position and current of the wire are reflected across the dashed line, the magnetic field it generates would be **reflected and reversed**.
Appendix II: Continuous Current Distribution

\[ \vec{j}(\vec{r}) \] - local current density

The flux of the current vector field

\[ I_{encl} = \int_{surface} \vec{j}(r) \cdot d\vec{a} \]

\[ \oint_{loop} \vec{B}(r) \cdot d\vec{l} = \mu_0 \int_{surface} \vec{j}(r) \cdot d\vec{a} \]

There are infinitely many possible surfaces that have the loop as their border. Which of those surfaces is to be chosen? It does not matter; for all these surfaces \( \int_{surface} \vec{j}(r) \cdot d\vec{a} \) would be the same (due to the continuity equation for charge).
Appendix III: Field of a Toroidal Solenoid

Symmetries of the current distribution:
- rotations around the center in the plane of the toroid;
- reflections across any plane perpendicular to the plane of the toroid and going through its center.

the radial component of $B$ is zero; $B$ can depend only on the distance from the axis.

Amperian loops: circles centered at the toroid’s axis

Loop 1 $B(r)2\pi r = 0$ $B(r) = 0$ $r < r_1$

Loop 2 $B(r)2\pi r = \mu_0 IN$ $B(r) = \frac{\mu_0 IN}{2\pi r}$ $r_1 < r < r_2$

Loop 3 $B(r)2\pi r = 0$ $B(r) = 0$ $r > r_2$