Lecture 15. Magnetic Fields of Moving Charges and Currents

Outline:

- Magnetic Field of a Moving Charge.
- Magnetic Field of Currents.
- Interaction between Two Wires with Current.

Lecture 14:

- Hall Effect.
- Magnetic Force on a Wire Segment.
- Torque on a Current-Carrying Loop.
A circular loop of wire carries a constant current. If the loop is placed in a region of uniform magnetic field, the net magnetic force on the loop is

A. perpendicular to the plane of the loop, in a direction given by a right-hand rule.
B. perpendicular to the plane of the loop, in a direction given by a left-hand rule.
C. in the same plane as the loop.
D. zero.
E. The answer depends on the magnitude and direction of the current and on the magnitude and direction of the magnetic field.
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C. in the same plane as the loop.

D. zero.

E. The answer depends on the magnitude and direction of the current and on the magnitude and direction of the magnetic field.
\[ \mu = NAI \]
\[ \tau = \mu \times B \]
What is the torque on the current loop shown?

A. $4.5\pi \times 10^{-2}$ Nm into board
B. $4.5\pi \times 10^{-2}$ Nm out of board
C. $1.5\pi \times 10^{-2}$ Nm up
D. $1.5\pi \times 10^{-2}$ Nm down
E. Zero

$\tau = 3\pi (0.05) \cdot 2 \cdot 5 \cdot 1.2$

$= 4.5 \cdot 10^{-2}$ Nm

Direction: up $\times$ right = into board
Which of the following statements is FALSE?

a) The magnetic torque on a current-carrying coil of wire is larger when the magnetic field is perpendicular to the plane of the coil than when the magnetic field is in the plane of the coil.

b) The magnetic force on a charged particle moving along a magnetic field line is zero.

c) The magnetic force does zero work on a charged particle moving in a magnetic field.

d) A current-carrying planar loop of wire in a constant, uniform magnetic field has zero net magnetic force on it.

e) The net magnetic flux through any closed surface is zero.
Iclicker Question

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e) The net magnetic flux through any closed surface is zero.
Positive charge moving with a constant velocity $\mathbf{v} \ll \mathbf{c}$:

- $\mathbf{c}$ – the speed of light

$$B(r) = \frac{\mu_0}{4\pi} \frac{qv \times r}{r^2} = \frac{\mu_0}{4\pi} \frac{q}{v} \frac{r}{r^2}$$

- the charge is at the origin at this moment

$\mu_0 = 4\pi \times 10^{-7} \text{ Ns}^2 / \text{C}^2$

“permeability of the vacuum”
A positive point charge is moving directly toward point $P$. The magnetic field that the point charge produces at point $P$

A. points from the charge toward point $P$.
B. points from point $P$ toward the charge.
C. is perpendicular to the line from the point charge to point $P$.
D. is zero.
E. The answer depends on the speed of the point charge.

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$B(r) = \mu_0 \frac{q v \times r}{4\pi r^2}$
Iclicker Question

Two positive point charges move side by side in the same direction with the same velocity.

What is the direction of the magnetic force that the upper point charge exerts on the lower one?

A. toward the upper point charge (the force is attractive)
B. away from the upper point charge (the force is repulsive)
C. in the direction of the velocity
D. opposite to the direction of the velocity
E. none of the above

\[ B(r) = \frac{\mu_0}{4\pi} qv \times r / r^3 \]

\[ F = qv \times B \]
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\[ B(r) = \frac{\mu_0}{4\pi} \frac{qv \times r}{r^3} \]

\[ F = qv \times B \]
Magnetic Field of Currents

Superposition:

\[ B(r) = \frac{\mu_0}{4\pi} \oint I dl \times r / r^2 \]

the current segment is at the origin

The charge of carriers in a wire segment \( A dl \):

\[ q \downarrow seg = \sum q \downarrow i = ne A dl \]

\[ I = ne v \downarrow d A \quad q \downarrow seg v \downarrow d = I dl \]

\[ B(r) = \frac{\mu_0}{4\pi} I dl \times r / r^2 \]

- proportional to \( 1/r^2 \), as the electric field of a point charge

The magnetic field of a wire loop with current:

\[ B(r) = \frac{\mu_0}{4\pi} I \oint dl \times (r - r') / |r - r'|^2 \]

“Biot-Savart Law”
In the figure, an irregular loop of wire carrying a current lies in the plane of the paper. Suppose that the loop is distorted into some other shape while remaining in the same plane. Point P is still within the loop. Which of the following is a TRUE statement concerning this situation?

a) The magnetic field at point P will always lie in the plane of the paper.

b) It is possible that the magnetic field at point P is zero.

c) The magnetic field at point P will not change in magnitude when the loop is distorted.

d) The magnetic field at point P will not change in direction when the loop is distorted.

e) None of the other statements are true.

\[ B(r) = \mu_0 \frac{I \Phi}{4\pi} \int \frac{dl \times (r - r')}{|r - r'|^3} \]
In the figure, an irregular loop of wire carrying a current lies in the plane of the paper. Suppose that the loop is distorted into some other shape while remaining in the same plane. Point P is still within the loop. Which of the following is a TRUE statement concerning this situation?

a) The magnetic field at point P will always lie in the plane of the paper.
b) It is possible that the magnetic field at point P is zero.
c) The magnetic field at point P will not change in magnitude when the loop is distorted.

**d) The magnetic field at point P will not change in direction when the loop is distorted.**
e) None of the other statements are true.
Find the magnetic field at the center of a circular loop with current.

\[ B(r) = \frac{\mu_0}{4\pi} \frac{I}{|r - r'|} \times (r - r')/|r - r'| \]

Let’s place the origin at the center of the loop \( (r=0) \).

\[ |r - r'| = R \]

\[ dl = R \, d\alpha, \quad 0 \leq \alpha \leq 2\pi \]

\[ B(0) = \frac{\mu_0}{4\pi} \frac{I}{R} \]

\[ = \frac{\mu_0}{4\pi} \frac{I}{R} \int_0^{2\pi} \frac{R}{R^2} d\alpha = \frac{\mu_0 I}{2R} \]

The field at the center of the loop:

\[ B(0) = \frac{\mu_0 I}{2R} \]

For \( N \) turns with current, the field is \( N \) times stronger.
Magnetic Field Inside a Solenoid

\[ B (r) = \mu_0 \frac{1}{4\pi} \oint \mathbf{I} \cdot \mathbf{dl} \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \]

Consider a long solenoid with \( n = N/L \) windings per unit length:

Find the field from Biot-Savart (above), or from Ampere’s law (next lecture).

The field at the center of the solenoid:

\[ B = \mu_0 nI \]

The field at the ends of the solenoid (on axis):

\[ B = \mu_0 nI/2 \]
A wire consists of two straight sections with a semicircular section between them. If current flows in the wire as shown, what is the direction of the magnetic field at $P$ due to the current?

A. to the right
B. to the left
C. out of the plane of the figure
D. into the plane of the figure
E. misleading question — the magnetic field at $P$ is zero
Iclicker Question

A wire consists of two straight sections with a semicircular section between them. If current flows in the wire as shown, what is the direction of the magnetic field at P due to the current?

A. to the right
B. to the left
C. out of the plane of the figure
D. into the plane of the figure
E. misleading question — the magnetic field at P is zero

\[ B(r) = \mu_0 I \frac{\hat{z} \cdot \mathbf{l} \times (r - r')}{|r - r'|} \]
Find $B$ at a distance $r$ from a straight wire segment carrying $I$.

Let's choose the origin at the point where we measure $B(r=0)$:

$$B(r=0) = \mu_0 /4\pi I \int_{\alpha_1}^{\alpha_2} \mathbf{dl} \times (-r') /|r'|$$

Express $dl$ and $r'$ in terms of $r$ and $\alpha$:

$$l = r \tan \alpha$$

$$dl = r_1 /\cos^2 \alpha \, d\alpha$$

$$r' = r /\cos \alpha$$

$$|dl \times (-r')| = dl \cdot r' \cdot \sin(90+\alpha) = dl \cdot r' \cdot \cos(\alpha) = rd\alpha /\cos \alpha \cdot r = r \int_{\alpha_2}^{\alpha_1} d\alpha /\cos \alpha$$

Example: Magnetic field of a current flowing around a square loop, at the center of the loop ($\alpha_1 = -\pi/4$, $\alpha_2 = \pi/4$):

$$B(r) = \sum i \mu_0 I /4\pi r \int_{\alpha_1}^{\alpha_2} \cos \alpha$$

$$B(r) = \mu_0 I /\sqrt{2} \pi a$$
Magnetic field of an infinitely long straight wire

\[ B(r) = \mu_0 \frac{I}{4\pi r} (\sin \alpha_2 - \sin \alpha_1) \]

Infinitely long straight wire: \( \alpha_2 = \pi/2, \alpha_1 = -\pi/2 \)

\[ B(r) = \mu_0 \frac{I}{2\pi r} \]

(the \( r \)-dependence resembles that for \( E(r) \) of an infinite charged wire)
Consider a wire bent in the hairpin shape. The wire carries a current $I$. What is the magnitude of the magnetic field at point $a$?

Superposition: the field at point $a$ is a superposition of three $B$ fields produced by the current in the semi-circle and two straight wires.

**Semi-circle:**

$$B(r) = 1/2 \mu_0 I/2R = \mu_0 I/4R$$

(1/2 of the field of a circular loop)

$$B(r) = \mu_0 I/4\pi R \int_{\alpha_1}^{\alpha_2} = -\pi/2 \cos \alpha d\alpha = \mu_0 I/4\pi R [\sin0 - \sin(\pi/2)] = \mu_0 I/4R$$

**Top wire:**

$$(1/2 of the field of an infinite wire)$$

$$B(r) = \mu_0 I/4\pi R \int_{\alpha_1}^{\alpha_2} = \pi/2 \cos \alpha d\alpha = \mu_0 I/4\pi R$$

**Bottom wire:**

$$B(r) = \mu_0 I/4R + \mu_0 I/2\pi R = \mu_0 I/4R (1+2/\pi)$$
Interaction between Two Wires with Current

The magnetic field due to $I\downarrow 1$ at the position of the wire with $I\downarrow 2$

$$B(r) = \mu_0 \frac{I\downarrow 1}{2\pi r}$$

Force per unit length:

$$F/L = I\downarrow 1 B = \mu_0 \frac{I\downarrow 1 I\downarrow 2}{2\pi r}$$

Parallel (anti-parallel) currents attract (repel) each other.

SI unit and definition for electric current: The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross section, and placed 1 meter apart in vacuum, would produce between these conductors a force equal to $2\times10^{-7}$ Newton per meter of length.
Next time: Lecture 16: Ampere’s Law.
§§ 28.6- 28.8
The long, straight wire $AB$ carries a 10-A current as shown. The rectangular loop has long edges parallel to $AB$ and carries a clockwise 5-A current.

What is the magnitude and direction of the net magnetic force that the straight wire $AB$ exerts on the loop?

\[ F_{\text{tot}} = L \mu_0 |I\downarrow_1| |I\downarrow_2| /2\pi (1/r\downarrow_1 - 1/r\downarrow_2) \]

A. $\mu_0 /2\pi 5000$ N to the right

B. $\mu_0 /2\pi 5000$ N to the left

C. $\mu_0 /2\pi 2000$ N upward (toward $AB$)

D. $\mu_0 /2\pi 2000$ N downward (away from $AB$)

E. misleading question — the net magnetic force is zero
The long, straight wire $AB$ carries a 14-A current as shown. The rectangular loop has long edges parallel to $AB$ and carries a clockwise 5-A current.

What is the magnitude and direction of the net magnetic force that the straight wire $AB$ exerts on the loop?

$$F_{\text{tot}} = L\mu_0 I_{\downarrow 1} I_{\downarrow 2} /2\pi (1/r_{\downarrow 1} - 1/r_{\downarrow 2})$$

A. $\mu_0 /2\pi 5000 \text{ N to the right}$

B. $\mu_0 /2\pi 5000 \text{ N to the left}$

C. $\mu_0 /2\pi 2000 \text{ N upward (toward } AB)$

D. $\mu_0 /2\pi 2000 \text{ N downward (away from } AB)$

E. misleading question — the net magnetic force is zero
**Electrostatics vs. Magnetostatics**

Elementary source of the static $E$ field: point charge (zero-dimensional object, scalar)

**Gauss’ Law:**
\[ \oint_{\text{surface}} E(\mathbf{r}) \cdot dA = \frac{q \text{ enclosed}}{\varepsilon_0} \]
- valid in electrodynamics!

\[ \oint_{\text{loop}} E(\mathbf{r}) \cdot dl = 0 \]
- valid in electrostatics only(!), will be modified in electrodynamics.

Elementary source of the static $B$ field: current segment (one-dimensional object, vector)

**Absence of magnetic monopoles:**
\[ \oint_{\text{surface}} B(\mathbf{r}) \cdot dA = 0 \]
- valid in electrodynamics!

\[ \oint_{\text{loop}} B(\mathbf{r}) \cdot dl = \mu_0 I \]
For a loop that doesn’t enclose any current, the circulation is 0.

In magnetostatics, currents are the only source of the non-zero circulation of $B$. This will be modified in electrodynamics.
Magnetic Force as a Relativistic Correction to the Electric Force

In general, electrical repulsion + magnetic attraction.

**Charges at rest** (the proton ref. frame), only electric force (repulsion):

\[ F_{\perp E} = \frac{1}{4\pi\varepsilon_0} \frac{q^2}{r^2} \]

**Charges in motion** (the lab. ref. frame, primed), both \( F_E \) and \( F_B \):

According to the special theory of relativity:

\[ F_{\perp \perp} = F_{\perp \perp}' = F_{\perp E}' + F_{\perp B}' \]

\[ \gamma = \frac{1}{\sqrt{1 - (v/c)^2}} \]

\[ F_{\perp E}' = \gamma F_{\perp E} \]

\[ B_{\perp \perp}' = \gamma B_{\perp \perp} \]

In the lab ref. frame:

\[ F_{\perp B}' = F_{\perp B}' - F_{\perp E}' = (\gamma - 1/\gamma) \frac{1}{4\pi\varepsilon_0} \frac{q^2}{r^2} \]

Magnetic force could be regarded as a relativistic correction to the electrical force.
Magnetic Field of Currents

The charge of carriers in a wire segment \( Adl \):
\[
q_{\text{seg}} = \sum i^\uparrow q_i = neAdl
\]

The current:
\[
I = JA = nev \downarrow dA
\]
\[
q_{\text{seg}} v \downarrow d = Idl
\]

\( J \)- current density, \( Idl \) - current segment

\[
B(r) = \frac{\mu_0}{4\pi} \frac{q_{\text{seg}} v \downarrow d \times (r - r')}{|r - r'|^3}
\]

Superposition:

Assuming the current segment is at the origin:

\[
B(r) = \frac{\mu_0}{4\pi} \frac{Idl \times r}{|r|^3} = \frac{\mu_0}{4\pi} \frac{Idl}{|r|^2}
\]

- proportional to \( 1/r^2 \), as an electric field of a point charge

The magnetic field of a wire loop with current:

The magnetic field at a distance \( r \) from a straight wire with current:

\[
B(r) = \frac{\mu_0}{4\pi} \frac{I \Phi^\uparrow dl \times (r - r')}{|r - r'|^3}
\]

\[
B(r) = \frac{\mu_0}{2\pi r}
\]

- proportional to \( 1/r \), as \( E(r) \) of an infinite charged wire

1820 – first observation of the magnetic field due to currents
A wire loop with current has a shape of a circle of radius $a$. Find the magnetic field at a distance $x$ from its center along the axis of symmetry.

$$B (r) = \frac{\mu_0}{4\pi} I \oint dl \times (r - r') / |r - r'|$$

Let’s place the origin at the center of the loop and introduce two angles: $\alpha$ and $\theta$.

$$|r - r'| = \sqrt{R^2 + x^2} \quad dl = R \, d\alpha, \quad 0 \leq \alpha \leq 2\pi \quad \cos \theta = \frac{R}{\sqrt{R^2 + x^2}}$$

$$B_{\perp x} (x) = \frac{\mu_0}{4\pi} I \int dl / R \sqrt{R^2 + x^2} \quad \cos \theta = \frac{\mu_0}{4\pi IR} (R \sqrt{R^2 + x^2})^{3/2} \int_0^{2\pi} \frac{\mu_0}{R} R \, d\alpha$$

$B_{\perp y}$ component is zero due to the symmetry.

The field at the center of the loop ($x=0$): $$B(0) = \frac{\mu_0}{2} I / 2R$$

For $N$ turns with current, the field is $N$ times stronger.