Mutual Inductance of a Double Solenoid

Description: Walks through calculation of mutual inductance for a double solenoid

Learning Goal:
To learn about mutual inductance from an example of a long solenoid with two windings.

To illustrate the calculation of mutual inductance it is helpful to consider the specific example of two solenoids that are wound on a common cylinder. We will take the cylinder to have radius $\rho$ and length $L$. Assume that the solenoid is much longer than its radius, so that its field can be determined from Ampère’s law throughout its entire length:

$$\oint B(\vec{r}) \cdot d\vec{l} = \mu_0 I_{encl}. $$

We will consider the field that arises from solenoid 1, which has $n_1$ turns per unit length. The magnetic field due to solenoid 1 passes (entirely, in this case) through solenoid 2, which has $n_2$ turns per unit length. Any change in magnetic flux from the field generated by solenoid 1 induces an EMF in solenoid 2 through Faraday’s law of induction,

$$\oint E(\vec{r}) \cdot d\vec{l} = -\frac{d}{dt} \Phi_M(t). $$

Part A

Consider first the generation of the magnetic field by the current $I_1(t)$ in solenoid 1. Within the solenoid (sufficiently far from its ends), what is the magnitude $B_1(t)$ of the magnetic field due to this current?

Express $B_1(t)$ in terms of $I_1(t)$, variables given in the introduction, and relevant constants.
ANSWER:

\[ B_1(t) = \mu_0 n_1 I_1(t) \]

Note that this field is independent of the radial position (the distance from the symmetry axis) for points inside the solenoid.

Part B

What is the flux \( \Phi_1(t) \) generated by solenoid 1’s magnetic field through a single turn of solenoid 2?

Express \( \Phi_1(t) \) in terms of \( B_1(t) \), quantities given in the introduction, and any needed constants.

**Hint 1.** The definition of flux

The definition of magnetic flux is \( \Phi_M = \oint \vec{B} \cdot d\vec{A} \). When the field is constant and perpendicular to the cross section, this is just the magnitude of the field times the area. What is the area \( A \) in this case?

Express \( A \) in terms of \( \rho \) and other constants.

ANSWER:

\[ A = \pi \rho^2 \]

ANSWER:

\[ \Phi_1(t) = B_1(t) \pi \rho^2 \]

Part C

Now find the electromotive force \( \mathcal{E}_2(t) \) induced across the entirety of solenoid 2 by the change in current in solenoid 1. Remember that both solenoids have length \( L \).

Express your answer in terms of \( dI_1(t)/dt, n_1, n_2 \), other parameters given in the introduction, and any relevant constants.

**Hint 1.** Find the flux from \( I_1(t) \)

In Part B, you found the flux through a single turn of solenoid 2, \( \Phi_1(t) \).

Now express \( \Phi_1(t) \) in terms of \( I_1(t) \), \( n_1 \), other quantities given in the introduction, and various constants such as \( \mu_0 \).

ANSWER:

\[ \Phi_1(t) = \mu_0 n_1 I_1(t) \pi \rho^2 \]
**Hint 2.** The EMF for the entire solenoid

The total EMF for a multiturn solenoid can be quite large (this is the principle of transformers that can produce kilovolts).

Express the total EMF through solenoid 2, $E_2(t)$, in terms of $\Phi_1(t)$, $d\Phi_1(t)/dt$, and other quantities given in the introduction.

**ANSWER:**

$$E_2(t) = \frac{-n_2Ld\Phi_1(t)}{dt}$$

**Hint 3.** Putting it together

You have now worked out three things:

1. the magnetic field from $I_1(t)$
2. the flux from this field
3. the EMF for the entire second solenoid

Put them together and you have the answer!

**ANSWER:**

$$E_2(t) = \frac{-n_2LdI_1(t)}{dt} \frac{1}{\mu_0n_1}\rho^2$$

**Part D**

This overall interaction is summarized using the symbol $M_{21}$ to indicate the *mutual inductance* between the two windings. Based on your previous two answers, which of the following formulas do you think is the correct one?

**ANSWER:**

- $E_2(t) = -M_{21}I_1(t)$
- $\frac{d}{dt} E_2(t) = -M_{21}I_1(t)$
- $E_2(t) = -M_{21} \frac{d}{dt} I_1(t)$
- $I_1(t) = -M_{21} \frac{d}{dt} E_2(t)$
- $I_1(t) = -M_{21} E_2(t)$
Mutual inductance indicates that a change in the current in solenoid 1 induces an electromotive force (EMF) in solenoid 2. When the double solenoid is thought of as a circuit element, this electromotive force is added into Kirchhoff's loop law. The constant of proportionality is the mutual inductance, denoted by $M_{21}$. The negative sign in the equation $\mathcal{E}_2(t) = -M_{21} \frac{d}{dt} I_1(t)$ comes from the negative sign in Faraday's law, and reflects Lenz's rule: The changing magnetic field due to solenoid 1 will induce a current in solenoid 2; this induced current will itself generate a magnetic field within solenoid 2, such that changes in the induced magnetic field oppose the changes in the magnetic field from solenoid 1.

**Part E**

Using the formula for the mutual inductance, $\mathcal{E}_2(t) = -M_{21} \frac{d}{dt} I_1(t)$, find $M_{21}$.

Express the mutual inductance $M_{21}$ in terms of $n_1$, $n_2$, quantities given in the introduction, and relevant physical constants.

**ANSWER:**

$$M_{21} = n_2 L \mu_0 n_1 \rho^2$$

**Part F**

Not surprisingly, if a current is sent through solenoid 2, it induces a voltage in solenoid 1. The mutual inductance in this case is denoted by $M_{12}$, the mutual inductance for voltage induced in solenoid 1 from current in solenoid 2. What is $M_{12}$?

Express the mutual inductance in terms of $n_1$, $n_2$, quantities given in the introduction, and relevant physical constants.

**Hint 1. A new symmetry**

This problem has a new type of symmetry: The labeling of the solenoids as 1 and 2 is arbitrary. If we relabeled the solenoids, the only change would be to interchange $n_1$ and $n_2$, the number of turns per length of the two solenoids. Therefore, the solution can be obtained simply by making this interchange in your answer to Part E.

**ANSWER:**

$$M_{12} = n_2 L \mu_0 n_1 \rho^2$$

This result that $M_{12}$ is equal to $M_{21}$ reflects the interchangeability of the two coils and applies even if the coils are only partially coupled (for example, if one coil is wound on a much larger cylinder or if only a fraction of the larger coil's flux is intercepted by the smaller coil). Because of this fact, the subscripts are generally omitted: There is only one mutual inductance between two coils, denoted by $M$: An EMF is generated in one coil by a change in current in the other coil.
Exercise 30.5

**Description:** Two toroidal solenoids are wound around the same form so that the magnetic field of one passes through the turns of the other. Solenoid 1 has \( N_1 \) turns and solenoid 2 has \( N_2 \) turns. When the current in solenoid 1 is \( i_1 \), the average flux through each... 

Two toroidal solenoids are wound around the same form so that the magnetic field of one passes through the turns of the other. Solenoid 1 has 750 turns and solenoid 2 has 400 turns. When the current in solenoid 1 is \( 6.50 \, \text{A} \), the average flux through each turn of solenoid 2 is \( 4.00 \times 10^{-2} \, \text{Wb} \).

**Part A**

What is the mutual inductance of the pair of solenoids?

**ANSWER:**

\[
M = \frac{N_2 \Phi_2}{i_1} = 2.46 \, \text{H}
\]

**Part B**

When the current in solenoid 2 is \( 2.60 \, \text{A} \), what is the average flux through each turn of solenoid 1?

**ANSWER:**

\[
\Phi = \frac{i_2 N_2 \Phi_2}{N_1} = 8.53 \times 10^{-3} \, \text{Wb}
\]

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**Self-Inductance of a Solenoid**

**Description:** Walks through calculation of self-inductance for a single solenoid with some discussion at the end.

**Learning Goal:**

To learn about self-inductance from the example of a long solenoid.

To explain self-inductance, it is helpful to consider the specific example of a long solenoid, as shown in the figure. This solenoid has only one winding, and so the EMF induced by its changing current appears across the solenoid itself. This contrasts with mutual inductance, where this voltage appears across a second coil wound on the same cylinder as the first.

Assume that the solenoid has radius \( R \), length \( L \) along the \( z \) axis, and is wound with \( n \) turns per unit length so that the total number of turns is equal to \( nL \). Assume that the solenoid is much longer than its radius.

As the current through the solenoid changes, the resulting magnetic flux through the solenoid will also change, and an electromotive force will be generated across the solenoid according to Faraday’s law of induction:

\[
\oint E \cdot dl = -\frac{d}{dt} \Phi_M(t)
\]

Faraday’s law implies the following relation between the self-induced EMF across the solenoid and the current passing through it:

\[
\mathcal{E}(t) = -L \frac{dI(t)}{dt}
\]
The "direction of the EMF" is determined with respect to the direction of positive current flow, and represents the direction of the induced electric field in the inductor. This is also the direction in which the "back-current" that the inductor tries to generate will flow.

Part A

Suppose that the current in the solenoid is $I(t)$. Within the solenoid, but far from its ends, what is the magnetic field $B(t)$ due to this current?

Express your answer in terms of $I(t)$, quantities given in the introduction, and relevant constants (such as $\mu_0$).

ANSWER:

$$B(t) = \mu_0 n I(t)$$

Note that this field is independent of the radial position (the distance from the axis of symmetry) as long as it is measured at a point well inside the solenoid.

Part B

What is the magnetic flux $\Phi_1(t)$ through a single turn of the solenoid?

Express your answer in terms of the magnetic field $B(t)$, quantities given in the introduction, and any needed constants.

ANSWER:

$$\Phi_1(t) = B(t) \pi R^2$$

Also accepted: $\mu_0 n I(t) \pi R^2$

Part C

Suppose that the current varies with time, so that $dI(t)/dt \neq 0$. Find the electromotive force $\mathcal{E}$ induced across the entire solenoid due to the change in current through the entire solenoid.
Express your answer in terms of $dI(t)/dt, n, Z,$ and $R$.

**Hint 1. Find the flux in terms of the current**

In Part B you found the flux $\Phi_1(t)$ through a single turn of the solenoid. Now find the flux $\Phi_{total}(t)$ through the entire solenoid.

Express your answer in terms of $I(t)$, other quantities given in the introduction, and various constants such as $\mu_0$.

**ANSWER:**

$$\Phi_{total}(t) = \mu_0 n^2 I(t) \pi R^2 Z$$

**Hint 2. Find the EMF for the entire solenoid**

You now have an expression for the magnetic flux that passes through the solenoid. From this, you should be able to derive an expression for the EMF in the solenoid. Suppose the total magnetic flux through the solenoid is $\Phi_{total}(t)$. What is the electromotive force $\mathcal{E}$ generated in the solenoid by the changing flux $\Phi_{total}(t)$?

Express your answer in terms of $d\Phi_{total}(t)/dt$ and its derivative, and other variables given in the introduction.

**ANSWER:**

$$\mathcal{E} = \frac{-d\Phi_{total}(t)}{dt}$$

**Hint 3. Putting it together**

You have now worked out three things:

1. the magnetic field from $I(t)$;
2. the flux from this field;
3. the EMF for the entire solenoid.

Put them together and you have the answer!

**ANSWER:**

$$\mathcal{E} = \frac{-n^2 Z dI(t)}{dt} \mu_0 \pi R^2$$

**Part D**

The self-inductance $L$ is related to the self-induced EMF $\mathcal{E}(t)$ by the equation $\mathcal{E}(t) = -L dI(t)/dt$. Find $L$ for a long solenoid. (Hint: The self-inductance $L$ will always be a positive quantity.)

Express the self-inductance in terms of the number of turns per length $n$, the physical dimensions $R$ and $Z$, and relevant constants.
This definition of the inductance is identical to another definition you may have encountered: $\Phi_M = LI$, where $\Phi_M$ is the magnetic flux due to a current $I$ in the inductor. To see the correspondence you should differentiate both sides of this equation with respect to time and use Faraday’s law, i.e., $\frac{d\Phi_M}{dt} = -\mathcal{E}$.

Now consider an inductor as a circuit element. Since we are now treating the inductor as a circuit element, we must discuss the voltage across it, not the EMF inside it. The important point is that the inductor is assumed to have no resistance. This means that the net electric field inside it must be zero when it is connected in a circuit. Otherwise, the current in it will become infinite. This means that the induced electric field $\vec{E}_n$ deposits charges on and around the inductor in such a way as to produce a nearly equal and opposite electric field $\vec{E}_c$ such that $\vec{E}_c + \vec{E}_n \to 0$. Kirchhoff’s loop law defines voltages only in terms of fields produced by charges (like $E_c$), not those produced by changing magnetic fields (like $E_n$). So if we wish to continue to use Kirchhoff’s loop law, we must continue to use this definition consistently. That is, we must define the voltage $V_{AB} = V_A - V_B = + \int_A^B \vec{E}_c \cdot d\vec{l}$ alone (note that the integral is from A to B rather than from B to A, hence the positive sign). So finally, $V_{AB} = \int_A^B \vec{E}_c \cdot d\vec{l} = \int_A^B -\vec{E}_n \cdot d\vec{l} = -\mathcal{E} = +L \frac{dI(t)}{dt}$, where we have used $\vec{E}_c + \vec{E}_n = 0$ and the definition of $\mathcal{E}$.

**Part E**

Which of the following statements is true about the inductor in the figure in the problem introduction, where $I(t)$ is the current through the wire?

**Hint 1. A fundamental inductance formula**

The self-inductance $L$ is related to the voltage $V = V_{\text{rm A}} - V_{\text{rm B}}$ across the inductor through the equation $V(t) = L \, dI(t)/dt$. Note that unlike the EMF, it does not have a minus sign.

**ANSWER:**

- If $I(t)$ is positive, the voltage at end A will necessarily be greater than that at end B.
- If $dI(t)/dt$ is positive, the voltage at end A will necessarily be greater than that at end B.
- If $I(t)$ is positive, the voltage at end A will necessarily be less than that at end B.
- If $dI(t)/dt$ is positive, the voltage at end A will necessarily be less than that at end B.

**Part F**

Now consider the effect that applying an additional voltage to the inductor will have on the current $I(t)$ already flowing through it (imagine that the voltage is applied to end A, while end B is grounded). Which one of the following statements is true?
**Hint 1. A fundamental inductance formula**

The self-inductance $L$ is related to the voltage $V = V_A - V_B$ across the inductor through the equation

$$V(t) = L \frac{dI(t)}{dt}.$$ 

Note that unlike the EMF, it does not have a minus sign. However, when applying Kirchhoff's rules and traversing the inductor in the direction of current flow, there will be a term $-L \frac{dI(t)}{dt}$, just as traversing a resistor gives a term $-I R$.

**ANSWER:**

If $V$ is positive, then $I(t)$ will necessarily be positive and $\frac{dI(t)}{dt}$ will be negative.

If $V$ is positive, then $I(t)$ will necessarily be negative and $\frac{dI(t)}{dt}$ will be negative.

If $V$ is positive, then $I(t)$ could be positive or negative while $\frac{dI(t)}{dt}$ will necessarily be negative.

If $V$ is positive, then $I(t)$ will necessarily be positive and $\frac{dI(t)}{dt}$ will be positive.

Note that when you apply Kirchhoff's rules and traverse the inductor in the direction of current flow, you are interested in

$$V_{BA} = V_B - V_A = -L \frac{dI(t)}{dt},$$

just as traversing a resistor gives a term $-I R$.

In sum: when an inductor is in a circuit and the current is changing, the changing magnetic field in the inductor produces an electric field. This field opposes the change in current, but at the same time deposits charge, producing yet another electric field. The net effect of these electric fields is that the current changes, but not abruptly. The "direction of the EMF" refers to the direction of the first, induced, electric field.

**± Basic Properties of Inductors**

**Description:** ± Includes Math Remediation. Practice determining units of inductance (presumed to be a new concept) and some basic conceptual questions about inductors.

**Learning Goal:**

To understand the units of inductance, the potential energy stored in an inductor, and some of the consequences of having inductance in a circuit.

After batteries, resistors, and capacitors, the most common elements in circuits are inductors. Inductors usually look like tightly wound coils of fine wire. Unlike capacitors, which produce a physical break in the circuit between the capacitor plates, the wire of an inductor provides an unbroken continuous path in which current can flow. When the current in a circuit is constant, an inductor acts essentially like a short circuit (i.e., a zero-resistance path). In reality, there is always at least a small amount of resistance in the windings of an inductor, a fact that is usually neglected in introductory discussions.

Recall that current flowing through a wire generates a magnetic field in the vicinity of the wire. If the wire is coiled, such as in a solenoid or an inductor, the magnetic field is strongest within the coil parallel to its axis. The magnetic field associated with current flowing through an inductor takes time to create, and time to eliminate when the current is turned off. When the current changes, an EMF is generated in the inductor, according to Faraday's law, that opposes the
change in current flow. Thus inductors provide electrical inertia to a circuit by reducing the rapidity of change in the current flow.

Inductance is usually denoted by $L$ and is measured in SI units of henries (also written henrys, and abbreviated $\text{rm}{\text{H}}$), named after Joseph Henry, a contemporary of Michael Faraday. The EMF $\text{cal}{(E)}{\text{EMF}}$ produced in a coil with inductance $L$ is, according to Faraday's law, given by

$$\text{cal}{(E)} = - \frac{L}{\Delta t} \Delta I.$$

Here $I/\Delta t$ characterizes the rate at which the current $I$ through the inductor is changing with time $t$.

**Part A**

Based on the equation given in the introduction, what are the units of inductance $L$ in terms of the units of $\text{EMF}$, $t$, and $I$ (respectively volts $V$, seconds $s$, and amperes $A$)?

**ANSWER:**

1. $1\text{H} = 1\left(\frac{V\cdot s}{A}\right)$
2. $1\text{H} = 1\left(\frac{V\cdot A}{s}\right)$
3. $1\text{H} = 1\left(\frac{s\cdot A}{V}\right)$
4. $1\text{H} = 1\left(\frac{1}{V\cdot s\cdot A}\right)$

**Part B**

What EMF is produced if a waffle iron that draws 2.5 amperes and has an inductance of 560 millihenries is suddenly unplugged, so the current drops to essentially zero in 0.015 seconds?

**Express your answer in volts to two significant figures.**

**ANSWER:**

93 $V$

The elevated voltage does not last long, but it can sometimes be large enough to produce a potentially dangerous spark. Moral: Be very careful when opening switches carrying current, especially if they are part of an inductive circuit!

Electrical potential energy $U$ is stored within an inductor in the form of a magnetic field when current is flowing through the inductor. In terms of the current $I$ and the inductance $L$, the stored electrical potential energy is given by

$$U = \frac{1}{2} LI^2.$$

**Part C**

Which of the following changes would increase the potential energy stored in an inductor by a factor of 5?

**Check all that apply.**
As indicated by the equation in the introduction to this part, the current flowing through an inductor is related to the amount of electrical potential energy stored in the inductor. If the current is graphed as a function of time, the slope of the curve indicates the rate at which potential energy in the inductor is increasing or decreasing. The rate at which energy changes over time is known as power.

Energy cannot be delivered to the inductor infinitely fast, nor can it be dissipated instantaneously in the form of heat or light by other circuit elements. Thus power can never be infinite. This implies that the curve of current versus time must be continuous. A graph is discontinuous when it contains a point at which the current jumps from one value to another without taking on all the values in between. When this happens, the slope of the curve at that location is infinite, which would imply infinite power in this case.

Part D

Which of the graphs illustrate how the current through an inductor might possibly change over time?

Type the numbers corresponding to the right answers in alphabetical order. Do not use commas. For instance, if you think that only graphs C and D are correct, type CD.

ANSWER:

ABC

All real circuits, even those that do not specifically have inductors in them, have at least a small amount of inductance, just as real inductors have a small amount of resistance in their windings. Circuit analysis in textbooks often assumes ideal batteries, resistors, capacitors, and inductors and hence neglects such subtle details of real circuits.
Description: An air-filled toroidal solenoid has N turns of wire, a mean radius of r, and a cross-sectional area of A. (a) If the current is I, calculate the magnetic field in the solenoid. (b) Calculate the self-inductance of the solenoid. (c) Calculate the...

An air-filled toroidal solenoid has 380 turns of wire, a mean radius of 14.5 $\text{cm}$, and a cross-sectional area of 4.50 $\text{cm}^2$.

Part A
If the current is 5.50 $\text{A}$, calculate the magnetic field in the solenoid.

ANSWER:

\[
B = \frac{4\pi \times 10^{-7}NI}{2\pi r} = 2.88 \times 10^{-3} \text{ T}
\]

Part B
Calculate the self-inductance of the solenoid.

ANSWER:

\[
L = \frac{4\pi \times 10^{-7}N^2A}{2\pi r} = 8.96 \times 10^{-5} \text{ H}
\]

Part C
Calculate the energy stored in the magnetic field.

ANSWER:

\[
U = \frac{1}{2} \frac{4\pi \times 10^{-7}N^2A}{2\pi r} I^2 = 1.36 \times 10^{-3} \text{ J}
\]

Part D
Calculate the energy density in the magnetic field.

ANSWER:

\[
u = \frac{\left(\frac{4\pi \times 10^{-7}NI}{2\pi r}\right)^2}{2 \cdot 4\pi \times 10^{-7}} = 3.31 \text{ J/m}^3
\]

Part E
Exercise 30.20

Description: It has been proposed to use large inductors as energy storage devices. (a) How much electrical energy is converted to light and thermal energy by a 160-W light bulb in one day? (b) If the amount of energy calculated in part A is stored in an inductor in which the current is 80.0 A, what is the inductance?

It has been proposed to use large inductors as energy storage devices.

Part A

How much electrical energy is converted to light and thermal energy by a 160-W light bulb in one day?

Express your answer with the appropriate units.

ANSWER:

\[ E = \frac{P \cdot 24 \cdot 3600}{10^6} = 13.8 \text{ MJ} \]

Part B

If the amount of energy calculated in part A is stored in an inductor in which the current is 80.0 A, what is the inductance?

Express your answer with the appropriate units.

ANSWER:

\[ L = \frac{2P \cdot 24 \cdot 3600}{(I)^2 \cdot 10^{-3}} = 4.32 \text{ kH} \]

Decay of Current in an L-R Circuit

Description: To understand the decay of current in an L-R circuit, derive the mathematical form of \( I(t) \), and find the time constant.

Learning Goal:

To understand the mathematics of current decay in an L-R circuit

A DC voltage source is connected to a resistor of resistance \( R \) and an inductor with inductance \( L \),
forming the circuit shown in the figure. For a long time before 
t=0, the switch has been in the position shown, so that a

current \( I_0 \) has been built up in the circuit by

the voltage source. At \( t=0 \) the switch is thrown to remove the

voltage source from the circuit. This problem concerns the

behavior of the current \( I(t) \) through the

inductor and the voltage \( V(t) \) across the

inductor at time \( t \) after \( t=0 \).

Part A

From \( t=0 \) onwards, what happens to the voltage \( V(t) \) across the inductor and the current \( I(t) \) through the inductor relative to their values prior to \( t=0 \)?

**Hint 1. What is the relation between current and voltage for the inductor?**

The current through an inductor cannot change discontinuously unless the voltage across the inductor is infinite. This follows from the fundamental equation relating voltage and current for an inductor. Find the voltage drop \( V_L(t) \) across the inductor in the direction of the current shown (clockwise).

**Express your answer in terms of \( L \), \( I(t) \), \( dI(t)/dt \), and any necessary constants.**

**ANSWER:**

\[
V_L(t) = -\frac{LdI(t)}{dt}
\]

**ANSWER:**

- \( V \) changes slowly and \( I \) changes abruptly.
- \( I \) changes slowly and \( V \) changes abruptly.
- Both change slowly.
- Both change abruptly.
After \( t=0 \), the battery no longer provides a voltage that drives current around the circuit. If the circuit did not contain an inductor, then the current would drop to zero immediately. However, inductors act to keep the current flowing. If the current starts to change, this causes an electromotive force (EMF) to form across the inductor that (by Lenz’s law) opposes the tendency for the current to change. Here, this causes the current through the inductor to persist for a while as it decays toward zero.

**Part B**

What is the differential equation satisfied by the current \( I(t) \) after time \( t=0 \)?

Express \( \frac{dI(t)}{dt} \) in terms of \( I(t) \), \( R \), and \( L \).

**Hint 1. Kirchhoff’s loop law**

Kirchhoff’s loop law tells us that the EMFs and potential differences around a circuit will sum to zero. If we follow the drops and increases in potential completely around a circuit, then we will return to the potential at the point at which we started. Use Kirchhoff’s loop law to write down a differential equation in \( I(t) \) for the sum \( \Sigma V_i \) of the voltage drops that arise from traversing the circuit in the direction of the current arrow (clockwise). The equation will have a term provided by the inductor and a term provided by the resistor, from Ohm’s law.

Express your answer in terms of \( I(t) \), its derivative \( \frac{dI(t)}{dt} \), \( R \), and \( L \).

**Hint 1. Sign of the voltage drop across the inductor**

The voltage drop across the inductor along the direction of current flow is \( -L \frac{dI(t)}{dt} \).

**ANSWER:**

\[
\Sigma V_i = -I(t)R - \frac{LdI(t)}{dt}
\]

**ANSWER:**

\[
\frac{dI(t)}{dt} = \frac{-I(t)R}{L}
\]

The minus sign in this equation tells us that the current is decreasing with time. The current is decaying. This is the case because the DC voltage source no longer acts to sustain the current.

**Part C**

What is the expression for \( I(t) \) obtained by solving the differential equation that \( I(t) \) satisfies after \( t=0 \)?

Express your answer in terms of the initial current \( I_0 \), as well as \( L \), \( R \).
Hint 1. Separation of variables

To solve the differential equation we may separate the variables and integrate. The variables in this equation are \( I \) and \( t \). To separate the variables, rearrange the equation so that all the \( I \) and \( \frac{dI}{dt} \) terms are on one side, and all the \( t \) and \( dt \) terms are on the other side. Complete the equation below in this format.

\[
\frac{dI}{I} = \frac{-R}{L} \, dt
\]

Hint 2. Integrating

Now integrate both sides of the differential equation. We know the limits of integration: \( I \) runs from \( I_0 \) to \( I(t) \), while \( t \) runs from 0 to \( t \). Recall also that \( \int_a^b \frac{dx}{x} = \ln \left( \frac{b}{a} \right) \).

ANSWER:

\[
I(t) = I_0 e^{-\frac{R}{L}t}
\]

Also accepted: \( I(t) = I_0 e^{-\frac{R}{L}t} \)

Part D

What is the time constant \( \tau \) of this circuit?

Express your answer in terms of \( L \) and \( R \)?

Hint 1. Definition of time constant

The time constant is the time, commonly represented by \( \tau \), that it takes for a variable to fall to \( 1/e \) (or about 0.37) of its initial value. The time constant is, as its name suggests, a constant. Hence a quantity that falls to 0.37 of its initial value after time \( \tau \) will fall to 0.37\( \cdot \)0.37\( \approx \)0.14 of its initial value after time \( 2\tau \). Writing this relation mathematically, for a quantity \( X \) that decays over time \( t \), we have \( X(t) = X_0 e^{-t/\tau} \), where \( X(0) = X_0 \) is the initial value (at time \( t = 0 \)).

Hint 2. The time constant for this circuit

Compare the mathematical description of the time constant with the expression for \( I(t) \) found earlier.

ANSWER:
Power in a Decaying R-L Circuit

**Description:** This problem calculates the power dissipated within both the resistor and inductor in an L-R circuit. Thus showing that the power dissipated by the resistor is equal to that coming out of the inductor.

Consider an R-L circuit with a DC voltage source, as shown in the figure.

This circuit has a current \( I_0 \) when \( t < 0 \). At \( t = 0 \) the switch is thrown removing the DC voltage source from the circuit. The current decays to \( I(t) \) at time \( t \).

**Part A**

What is the power, \( P_R(t) \), flowing into the resistor, \( R \), at time \( t \)?

Express your answer in terms of \( I(t) \) and \( R \).

**Hint 1. Ohm's Law**

State Ohm's law relating the potential difference, \( V(t) \), across the resistor to the current flowing through the resistor, \( I(t) \).

Express your answer in terms of \( V(t) \) and \( R \).

**ANSWER:**

\[
V(t) = I(t)R
\]

**Hint 2. Power Dissipated**

State the relationship between the power, \( P(t) \), dissipated by a voltage, \( V(t) \), driving a current, \( I(t) \).

Express your answer in terms of these quantities.
Part B

What is the power flowing into the inductor?

Express your answer as a function of $R$ and $I(t)$.

**Hint 1. Energy in an Inductor**

What is the energy, $U(t)$, stored within an inductor?

Express your answer in terms of $L$ and $I(t)$.

**ANSWER:**

$$U(t) = \frac{1}{2} LI(t)^2$$

Compare this with the energy stored in a capacitor, $U=\frac{1}{2} CV^2$.

**Hint 2. Equation for $I(t)$**

What is the exponential equation for the decaying current, $I(t)$?

Express your answer in terms of the initial current $I_0$, $t$, $R$, and $L$.

**ANSWER:**

$$I(t) = I_0 e^{-\frac{tR}{L}}$$

**Hint 3. Power into the Inductor**

Power is the rate of change of energy with respect to time. To find the power input to the inductor, write the energy as a function of time and differentiate with respect to time. Note that $\frac{dI(t)}{dt} \propto I(t)$.

Also accepted:

$$RI_0^2 e^{-\frac{2\pi}{L}}$$
ANSWER:

\[
\text{P}_{L}(t) = -RI(t)^2
\]

Also accepted: \(-RI_0^2e^{-\frac{2t}{L}}\)

**Part C**

Compare the two equations for power dissipated within the resistor and inductor. Which of the following conclusions about the shift of energy within the circuit can be made?

ANSWER:

- Power comes out of the inductor and is dissipated by the resistor
- Power is dissipated by both the inductor and the resistor
- Power comes out of both the inductor and the resistor
- Power comes out of the resistor and is dissipated by the inductor

The difference in sign between the power dissipated by the inductor and that by the resistor tells us that one is providing the power which is dissipated by the other. The inductor provides power from its magnetic field which is subsequently dissipated within the resistor as heat for example.

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**Energy within an L-C Circuit**

**Description:** This problem analyzes an L-C circuit with no voltage source. The question revises the equations for energy within the magnetic field of an inductor and the electric field of a capacitor. These are used to calculate the total energy within the circuit. The student is assumed to know some things about L-C circuits.

Consider an L-C circuit with capacitance \(C\), inductance \(L\), and no voltage source, as shown in the figure. As a function of time, the charge on the capacitor is \(Q(t)\) and the current through the inductor is \(I(t)\). Assume that the circuit has no resistance and that at one time the capacitor was charged.
Part A

As a function of time, what is the energy \( U_{\text{L}}(t) \) stored in the inductor?

Express your answer in terms of \( L \) and \( I(t) \).

\[
U_{\text{L}}(t) = \frac{1}{2} LI(t)^2
\]

An inductor stores energy in the magnetic field inside its coils.

Part B

As a function of time, what is the energy \( U_{\text{C}}(t) \) stored within the capacitor?

Express your answer in terms of \( C \) and \( Q(t) \).

\[
U_{\text{C}}(t) = \frac{1}{2} Q(t)^2
\]

A capacitor stores energy within the electric field between its plates (conductors).

Part C

What is the total energy \( U_{\text{total}} \) stored in the circuit?

Express your answer in terms of the maximum current \( I_0 \) and inductance \( L \).

\[
U_{\text{total}} = \frac{1}{2} LI_0^2
\]

Hint 1. Charge and current as functions of time

In order to produce a solution that does not contain terms involving the capacitance and the charge, we need to express charge and current as functions of time. An \( L-C \) circuit will undergo resonance, with the current varying sinusoidally, where \( I(t) = I_0 \sin(\omega t) \). Integrate \(-I(t)\) to obtain a similar expression for \( Q(t) \).

Express your answer in terms of \( I_0 \), \( t \), and \( \omega \).

\[
Q(t) = \frac{I_0}{\omega} \cos(\omega t)
\]

Hint 2. Find the resonance frequency

We also require an expression for the angular frequency of oscillation of the circuit \( \omega \).
What is the angular frequency of an $L$-$C$ circuit?

Express your answer in terms of $L$ and $C$.

ANSWER:

\[
\omega = \sqrt{\frac{1}{LC}}
\]

**Hint 3. Summing the contributions**

Combine the expressions we have obtained, $I(t) = -I_0 \sin(\omega t)$ and $Q(t) = \frac{I_0}{\omega} \cos(\omega t)$, and the equation for $\omega$ to express the total energy in terms of the inductance $L$ and the maximum current $I_0$. You will need the trigonometric relation $\cos^2(x) + \sin^2(x) = 1$.

ANSWER:

\[
U_{\text{total}} = \frac{1}{2} L I_0^2
\]

We could have eliminated the inductance from this expression instead of the capacitance to find that $U_{\text{total}} = \frac{Q_0^2}{2C}$. This total energy remains constant; however, the location of the energy changes. We can tell from the $\cos^2$ and $\sin^2$ terms in the inductor's and the capacitor's energies that as time passes, the energy moves back and forth between the inductor and the capacitor.

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**± L-R-C AC Circuits**

**Description:** ± Includes Math Remediation. Reactance and impedance calculations with LRC ac circuits.

**Learning Goal:**

To understand basic calculations involving $L$-$R$-$C$ ac circuits.

Because the currents and voltages vary, ac circuits are more complex than dc circuits. Consider a circuit consisting of a resistor of resistance $R$, an inductor of inductance $L$, and a capacitor of capacitance $C$ connected in series to an ac power source. As with all circuit components connected in series, the same current flows through each of these elements. This is an ac circuit, so the current is changing with time. The current $i(t)$ at a time $t$ can be found using the relationship: $i = I \cos(\omega t)$, where $I$ is the maximum current and $\omega = 2\pi f$ is the frequency of the current source.

The relationship between the current and voltage in an ac circuit works according to Ohm's law. Consider just the resistor in the circuit. Because the current changes in time, the voltage across the resistor $v_R$ also changes. However, both $i(t)$ and $v_R$ will be at a maximum at the same time. The maximum voltage across the resistor is given by $V_R = I R$. 

Recall that an inductor is designed to oppose any change in current in the circuit. Although an inductor has no resistance, there is a potential difference $v_L$ across the ends of the inductor. Unlike the case for the resistor, $i$ and $v_L$ do not reach maximum values at the same time. The voltage reaches a maximum before the current.

The maximum potential difference across the inductor is

$$V_L = I \omega L.$$  

By defining the quantity $\omega L$ as the inductive reactance $X_L$, $V_L$ can be rewritten as $V_L = I X_L$. This equation is similar to Ohm's law.

**Part A**

An $L$-$R$-$C$ circuit, operating at 60 Hz, has an inductor with an inductance of $1.53 \times 10^{-3}$ H, a capacitance of $1.67 \times 10^{-2}$ F, and a resistance of 0.329 $\Omega$. What is the inductive reactance of this circuit?

Enter your answer numerically in ohms.

**Hint 1. Find $\omega$**

The frequency $f$ is 60 Hz. You must first find $\omega$ from the frequency using $\omega = 2\pi f$. What is $\omega$?

Enter your answer numerically in inverse seconds.

**ANSWER:**

$$\omega = 377 \text{ s}^{-1}$$

ANSWER:

$$X_L = 0.577 \text{ } \Omega$$
Because $X_L$ is proportional to the frequency, low-frequency ac currents pass through an inductor more easily than high-frequency ac currents. Hence circuits containing inductors are often used as filters that allow low frequencies but not high frequencies to pass. These filters are called low-pass filters.

A capacitor is designed to store energy by allowing charge to build up on its plates. Although a capacitor has no resistance in an ac circuit, there is a potential difference $v_C$ across the plates of the capacitor. The maximum values of $i$ and $v_C$ do not occur at the same time. The voltage reaches a maximum after the current.

The maximum potential difference across the inductor is

$$V_C = \frac{|i|}{\omega C}.$$  

By defining the quantity $1/\omega C$ as the capacitive reactance $X_C$, $V_C$ can be rewritten as $V_C = I X_C$. As with the case for the inductor, this equation is similar to Ohm's law.

**Part B**

What is the capacitive reactance of the circuit in Part A?

Enter your answer numerically in ohms.

**ANSWER:**

$X_C = 0.159 \ \Omega$

Because $X_C$ is inversely proportional to the frequency, high-frequency ac currents pass through a capacitor more easily than low-frequency ac currents. Hence circuits containing capacitors are often used as filters that allow high frequencies but not low frequencies to pass. These filters are called high-pass filters.

The capacitive reactance, the inductive reactance, and the resistance of the circuit can be combined to give the impedance $Z$ of the circuit. The impedance is a measure of the total reactance and resistance of the circuit and is similar to the equivalent resistance that can be found from various resistors in a dc circuit. Because $v_L$ and $v_C$ do not reach maximum values at the same time as $i$, the impedance is not found by adding $R$, $X_L$, and $X_C$. Instead, the impedance is found using

$$Z = \sqrt{R^2 + (X_L - X_C)^2}.$$  

The analogy to Ohm's law is then

$$V = I Z.$$  

**Part C**

What is the total impedance of the circuit in Parts A and B?

Enter your answer numerically in ohms.

**ANSWER:**
In ac circuits, \(I\) and \(V\) are not measured directly. Instead, ac ammeters and voltmeters are designed to measure the root-mean-square values of \(I\) and \(V\):

\[
I_{\text{rms}} = \frac{I}{\sqrt{2}} \quad \text{and} \quad V_{\text{rms}} = \frac{V}{\sqrt{2}},
\]

which can be used in the equation \(V = I Z\) to yield

\[
V_{\text{rms}} = I_{\text{rms}} Z.
\]

**Part D**

If this circuit were connected to a standard 120 V ac outlet, what would the rms current in the circuit be?

**Enter your answer numerically in amperes.**

**ANSWER:**

\[
I_{\text{rms}} = 226 \ \text{A}
\]

The current is high because the total impedance is relatively low. Actually, plugging such a circuit into a 120-V outlet would most likely burn out the circuit elements.

A frequent application of \(L-R-C\) ac circuits is the tuning mechanism in a radio. The \(L-R-C\) ac circuit will have a resonant frequency that depends on both the inductance and capacitance of the circuit according to the formula

\[
f_0 = \frac{1}{2 \pi \sqrt{LC}}.
\]

This is the frequency at which the impedance is the smallest, which causes the largest current to appear in the circuit for a given \(V_{\text{rms}}\). The radio picks up this resonant frequency and suppresses signals at other frequencies. A variable capacitor in this circuit causes the resonant frequency of the circuit to change. When you tune the radio you are adjusting the value of the capacitance in the circuit and hence the resonant frequency.

**Part E**

To see whether the \(L-R-C\) ac circuit from Part A would be suitable for a tuner in a radio, find the resonant frequency of this circuit.

**Enter your answer numerically in hertz.**

**ANSWER:**

\[
f_0 = 31.5 \ \text{Hz}
\]

This frequency does not correspond to either the standard AM or FM band.

---

**Self-Inductance of a Coaxial Cable**

\[
L = 0.532 \ \text{mH}
\]

\[
I_{\text{rms}} = 226 \ \text{A}
\]

\[
f_0 = 31.5 \ \text{Hz}
\]
Description: Find the self-inductance per unit length of a coaxial cable.

A coaxial cable consists of alternating coaxial cylinders of conducting and insulating material. Coaxial cabling is the primary type of cabling used by the cable television industry and is also widely used for computer networks such as Ethernet, on account of its superior ability to transmit large volumes of electrical signal with minimum distortion. Like all other kinds of cables, however, coaxial cables also have some self-inductance that has undesirable effects, such as producing some distortion and heating.

Part A

Consider a long coaxial cable made of two coaxial cylindrical conductors that carry equal currents $I$ in opposite directions (see figure). The inner cylinder is a small solid conductor of radius $a$. The outer cylinder is a thin walled conductor of outer radius $b$, electrically insulated from the inner conductor. Calculate the self-inductance per unit length $\frac{L}{l}$ of this coaxial cable. ($L$ is the inductance of part of the cable and $l$ is the length of that part.)

Due to what is known as the "skin effect", the current $I$ flows down the (outer) surface of the inner conducting cylinder and back along the outer surface of the outer conducting cylinder. However, you may ignore the thickness of the outer cylinder.

Express your answer in terms of some or all the variables $I$, $a$, $b$, and $\mu_0$, the permeability of free space.

Hint 1. How to approach the problem

The self-inductance of the cable can be found if both the current and the magnetic flux through the cable are known. Using Ampère's law, you can calculate the magnetic field due to the cable; you will find that there is a magnetic field in the region between the two conductors. Thus, calculate the magnetic flux per unit length through the region between the two cylinders. Since the current in the cable is given, now you should have enough information to calculate the self-inductance of the cable.

Hint 2. Find the magnetic field inside the inner cylinder

Consider the inner conducting cylinder. It carries a current $I$ along its outer surface. What is the magnitude of the magnetic field $B_1$ at a distance $r$ from the axis of the cylinder, in the region inside the cylinder?

Express your answer in terms of some or all of the variables $I$, $a$, $b$, and $\mu_0$, the permeability of free space.

Hint 1. Ampère's law

Ampère's law states that the line integral of the magnetic field $\mathbf{B}$ calculated around any closed path is given by

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}},$$

where $I_{\text{enc}}$ is the permeability of free space, and $I_{\text{enc}}$ is
the net current through the area enclosed by the path. The best loop to use is a circle centered on the cylinders’ axis, because the magnetic field will be tangent to the loop at all points, making the dot product easier to compute.

**Answer:**

\[ \text{Hint 3. Find the magnetic field between the cylinders} \]

Now consider both conducting cylinders, each carrying equal currents \( I \) in opposite directions. Find the magnitude of the magnetic field \( B_2 \) at a distance \( r \) from the axis of the cable in the region between the two cylinders.

Express your answer in terms of some or all of the variables \( I \), \( r \), and \( \mu_0 \), the permeability of free space.

**Hint 1. Ampère's law**

Ampère’s law states that the line integral of the magnetic field \( \vec{B} \) calculated around any closed path is given by

\[
\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{\text{encl}}
\]

where \( \mu_0 \) is the permeability of free space, and \( I_{\text{encl}} \) is the net current through the area enclosed by the path. The best loop to use is a circle centered on the cylinders’ axis, because the magnetic field will be tangent to the loop at all points, making the dot product easier to compute.

**Answer:**

\[ \text{Hint 4. Find the magnetic field outside the cable} \]

Finally, find the magnitude of the magnetic field \( B_3 \) at a distance \( r \) from the axis of the cable in the region outside the cable. Again, assume that a current \( I \) flows along the surface of the inner cylinder and flows back on the surface of the outer cylinder.

Express your answer in terms of some or all of the variables \( I \), \( r \), and \( \mu_0 \), the permeability of free space.

**Hint 1. Ampère's law**

Ampère’s law states that the line integral of the magnetic field \( \vec{B} \) calculated around any closed path is given by
\[ \oint \vec{B} \cdot \vec{dl} = \mu_0 I_{\text{encl}} \],

where \( \mu_0 \) is the permeability of free space, and \( I_{\text{encl}} \) is the net current through the area enclosed by the path. The best loop to use is a circle centered on the cylinders’ axis, because the magnetic field will be tangent to the loop at all points, making the dot product easier to compute.

**ANSWER:**

\[ \text{\texttip{B_{\text{3}}}{B_3}} = 0 \]

The fields due to the opposite currents cancel each other out.

**Hint 5. Find the magnetic flux in the cable**

Find the magnetic flux per unit length \( \frac{\Phi_B}{l} \) in the cable due to the currents \( I \). Let \( a \) be the radius of the inner cylinder and \( b \) that of the outer cylinder.

Express your answer in terms of some or all the variables \( I, a, b, \) and \( \mu_0 \), the permeability of free space.

**Hint 1. How to approach the problem**

The magnetic field of the coaxial cable exists only in the region between the two cylinders. Therefore, you need to calculate the magnetic flux per unit length through this portion of the cable. To do that, think about the direction of the magnetic field (using the right-hand rule) and what area element to use.

**Hint 2. How to calculate the magnetic flux**

Given a finite area \( A \) in a magnetic field \( \vec{B} \), the magnetic flux \( \Phi_B \) through the area is defined by the following integral calculated over the area,

\[ \large{\Phi_B = \int \vec{B} \cdot d\vec{A}} \]

where \( d\vec{A} \) is an infinitesimal area element.

The magnetic field due to the cable exists only in the region between the cylinders and it is perpendicular to any plane containing the cable axis. Therefore, to calculate the magnetic flux it is most convenient to choose an area element of length \( \vert I \vert \) parallel to the cable axis, and of width \( dr \) lying in the plane containing the axis. The figure below shows such an area element with the cross section of the cable parallel to the cable axis depicted.

Such an area element is perpendicular to the magnetic field, making the dot product easy.
**ANSWER:**

\[
\frac{\Phi_B}{l} = \frac{L}{l} = \frac{\mu_0 I \ln \left( \frac{b}{a} \right)}{2\pi}
\]

**Hint 6. Formula for self-inductance**

The self inductance \( L \) of a circuit that carries a current \( i \) is given by

\[
L = \frac{\Phi_B}{i},
\]

where \( \Phi_B \) is the magnetic flux through the closed loop of the circuit.

**ANSWER:**

\[
\frac{C}{l} = \frac{2\pi \epsilon_0}{\ln \left( \frac{b}{a} \right)}
\]

The capacitance per unit length \( \frac{C}{l} \) of such a coaxial cable is \( \frac{2\pi \epsilon_0}{\ln \left( \frac{b}{a} \right)} \). So the product \( \frac{L}{l} \cdot \frac{C}{l} = \mu_0 \cdot \epsilon_0 = \frac{1}{c^2} \), where \( c \) is the speed of light in vacuum! As you might have guessed, this is not a coincidence, but a result that is quite generally true for such systems.