PHYSICS 203

6/18/2019
Example: Exploding Rocket

A rocket is fired vertically upward. At the instant it reaches an altitude of 1,000 m and a speed of 300 m/s, it explodes into three fragments having equal mass. One fragment continues to move upward with a speed of 450 m/s following the explosion. The second fragment has a speed of 240 m/s and is moving east right after the explosion. What is the velocity ($\vec{v}_3$) of the third fragment right after the explosion?

$$\vec{v}_3 = 150\hat{y} - 80\hat{x}$$
Lecture

• Review: Constant Angular Acceleration
• Review: Torque
• Rotational Dynamics
• Angular Momentum
Constant Angular Acceleration ($\alpha$)

The equations of motion for constant angular acceleration are the same as those for linear motion, with the substitution of the angular quantities for the linear ones.

\[ \theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2 \]

\[ \omega = \omega_0 + \alpha t \]

\[ \omega^2 = \omega_0^2 + 2\alpha \Delta \theta \]

\[ \omega_{average} = \frac{\omega + \omega_0}{2} \]

\[ x = x_0 + v_{x_0} t + \frac{1}{2} a_x t^2 \]

\[ v_x = v_{x_0} + a_x t \]

\[ v_x^2 = v_{x_0}^2 + 2a_x \Delta x \]

\[ v_{x_{average}} = \frac{v_x + v_{x_0}}{2} \]
Example: Rotating Wheel with Constant $\vec{\alpha}$

A wheel rotates with a constant angular acceleration of $3.50 \text{ rad/s}^2$. If the angular speed of the wheel is $2.00 \text{ rad/s}$ at $t=0 \text{ s}$ through what angular displacement (in degrees) does the wheel rotate in $2.00 \text{ s}$?

$$\Delta \theta = \frac{11.0 \text{ rad} \times 180^\circ}{\pi} = 630^\circ$$
A wheel rotates with a constant angular acceleration of $3.50 \, \text{rad/s}^2$. If the angular speed of the wheel is $2.00 \, \text{rad/s}$ at $t=0 \, \text{s}$ through what angular displacement (in degrees) does the wheel rotate in $2.00 \, \text{s}$?

Answer: $11.0 \, \text{rad} \ (630. \, \text{o})$

How many revolutions has the wheel turned during the time interval?

a) 11
b) 1.75
 c) 63
d) 3.5
e) None of the above
Vector Nature of Angular Quantities

- The angular velocity vector points along the axis of rotation; its direction is found using a right hand rule:
Vector (cross) Product

• The vector (or cross) product of two vectors is defined by:

\[
\hat{x} \times \hat{y} = \hat{z} \quad \hat{y} \times \hat{x} = -\hat{z} \\
\hat{y} \times \hat{z} = \hat{x} \quad \hat{z} \times \hat{y} = -\hat{x} \\
\hat{z} \times \hat{x} = \hat{y} \quad \hat{x} \times \hat{z} = -\hat{y}
\]

• zero otherwise.

• An important property of cross products is:

\[
|\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta
\]
Direction of Vector $\vec{C} = \vec{A} \times \vec{B}$

• The result of a vector cross product operation is itself a vector.

• The direction of the resulting vector is perpendicular to both vectors being cross-multiplied and given by the right-hand-rule.
Consider the wrench pivoted on the axis “O”.

If we apply a force $\vec{F}$ in a direction $\phi$ to the horizontal we can define the magnitude of the torque as:

$$\tau = rF \sin \phi$$

Torque should not be confused with force. Forces can cause a change in linear acceleration (i.e. Newton’s 2nd Law). Forces can also cause a change in rotation but the effectiveness of the forces depends on both the forces and the moment arms of the forces, in a combination that we call torque.
Torque

Magnitude of the torque is: \(|\vec{\tau}| = |\vec{r}| |\vec{F}| \sin \theta\)

If an object has zero net external torque then is said to be in "rotational" equilibrium:

\[\Sigma \vec{\tau}_{ext} = 0\]

Then if both \(\Sigma \vec{F}_{ext} = 0\) and \(\Sigma \vec{\tau}_{ext} = 0\) the object is in equilibrium.
Rotational Dynamics; Torque and Rotational Inertia

• Since \( \Sigma |\vec{F}| = ma \Rightarrow |\vec{\tau}| = mra \) (\( \vec{F} \) perpendicular to \( r \)), then for a point mass we can write: \( |\vec{\tau}| = mr^2 \alpha \)

• This is for a **single point mass**; what about an extended object?

• Because the angular acceleration is the same for the whole object, we have:

\[
\Sigma \tau = (\Sigma mr^2) \alpha
\]
Rotational Dynamics; Torque and Rotational Inertia

The quantity $I = \Sigma mr^2$ is called the moment of inertia of an object.

The distribution of mass with respect to the axis of rotation matters!
Newton’s 2\textsuperscript{nd} Law for Rotation

\[ \Sigma \tau = I \ddot{\alpha} \]

Where \( I = \Sigma m r^2 \) is the moment of inertia of a mass distribution about a rotating axis.

\( I \) is analogous to mass for rotating objects.
The rotational inertia of an object depends not only on its mass distribution but also the location of the axis of rotation—compare (f) and (g), for example.
Rotating Rod

A uniform rod of length $L$ and mass $M$ is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane. The rod is released from rest in the horizontal position. What is the initial angular acceleration of the rod and the initial linear acceleration of its right end?

Moment of inertia of rod: $I = \frac{1}{3} ML^2$
A hoop of mass $m$ and radius $r$, is released from rest and rolls without slipping down an incline of height $h$. At the same time, a block of mass $m$ is released from rest and at the same height “$h$” on a frictionless incline. Which object has the greatest kinetic energy at the bottom of the incline?

a) The hoop  
b) The block  
\[\text{c)} \quad \text{Both have the same KE} \]
\[KE_{\text{system}} = Mgh\]
\[\text{d)} \quad \text{Need more information}\]
A hoop of mass m and radius r, is released from rest and rolls without slipping down an incline of height h. At the same time, a block of mass m is released from rest and at the same height “h” on a frictionless incline. Which object reaches the bottom of the incline first?

a) The hoop

b) The block (Correct)

c) Both at the same time

d) Need more information
Consider an object as a collection of particles.

• Assuming it rotates about a fixed axis ‘z’ with angular speed $\omega$ then each particle has a kinetic energy determined by its mass and tangential speed.

• If the mass of the $i^{th}$ particle is $m_i$ and its tangential speed is $v_i$ then its kinetic energy is, as you’d expect:

$$K_i = \frac{1}{2} m_i v_i^2$$
Then, the total kinetic energy of the rotating object is the sum of the kinetic energies of all of its individual particles:

\[
KE_R = \sum KE_i = \sum \frac{1}{2} m_i v_i^2
\]

All particles have the same angular speed and that speed is related to each particle’s tangential speed: \( v_i = r_i \omega \)

So we can write the rotational kinetic energy of a rigid object rotating with angular speed \( \omega \) as:

\[
KE_R = \frac{1}{2} \left( \sum m_i r_i^2 \right) \omega^2
\]
Rotational Kinetic Energy

The quantity $\Sigma (m_i r_i^2)$ is, the moment of inertia of the object.

Then, the rotational kinetic energy of a rigid object can be expressed entirely in terms of rotational quantities as:

$$KE_R = \frac{1}{2} (\Sigma m_i r_i^2) \omega^2$$
Work and The Work-Kinetic Energy Theorem for Rotational Motion

The torque does work as it moves the wheel through an angle $\theta$:

$$\Delta W_R = \tau \Delta \theta$$

Work KE Theorem for Rotation:

$$\Sigma W_{R_{net}} = \frac{1}{2}I \omega_f^2 - \frac{1}{2}I \omega_i^2$$
Rotational Kinetic Energy

If an object has both rotational and translational motions then the total kinetic energy is given by:

$$KE = \frac{1}{2} Mv_{CM}^2 + \frac{1}{2} I \omega^2$$
Rotating Rod

A uniform rod of length L and mass M is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane. The rod is released from rest in the horizontal position. What is the angular speed of the rod when its center of mass is at the lowest point in its trajectory?

Moment of inertia of rod:  \( I = \frac{1}{3} ML^2 \)

\[ I\alpha = \frac{L}{2} Mg \Rightarrow \alpha = \frac{3}{2} \frac{g}{L} \]
Rotating Rod

A hoop of radius “r” and mass “m” is released from rest at the top of a ramp of height “h”. If it rolls without slipping, what is the speed of its center of mass at the bottom of the ramp? What is its angular velocity?

Moment of inertia of hoop: \( I = mr^2 \)

\[
mgh = \frac{1}{2}I\omega^2 + \frac{1}{2}mv_{cm}^2 \quad \Rightarrow \quad mgh = mv_{cm}^2
\]

\[
v_{cm} = \sqrt{gh}, \quad \omega = \frac{\sqrt{gh}}{r}
\]
Angular Momentum and Its Conservation

In analogy with linear momentum, we can define angular momentum \( L: L = I \omega \)

Since \( \alpha = \Delta \omega / \Delta t \) we can then write the total torque as being the rate of change of angular momentum.

\[
\Sigma \tau = \frac{\Delta L}{\Delta t}
\]

If the net torque on an object is zero, the total angular momentum is constant.

\[
\Sigma \tau_{\text{ext}} = 0 \Rightarrow \Delta L = 0 \Rightarrow I \omega = I_0 \omega_0 = \text{constant}
\]
Angular Momentum and Its Conservation

Therefore, systems that can change their rotational inertia through internal forces will also change their rate of rotation:
Vector Nature of Angular Quantities

Angular acceleration and angular momentum vectors also point along the axis of rotation.
A disk (Disk1) with a radius of 30.0 cm and a mass of 3.0 kg spins about its center with a constant angular speed of $\omega = 3.1 \, \text{rad/s}$. A second disk (Disk2) of mass $M$ and radius 15.0 cm is dropped on top of disk one and it sticks to it. If the speed of the two disk system is $\omega_f = 1.2 \, \text{rad/s}$, what is the mass of Disk2?

Moment of inertia of disk: $I = \frac{1}{2} MR^2$

$$\Delta L = 0 \Rightarrow I_{Disk1}\omega_i = (I_{Disk1} + I_{Disk2})\omega_f$$

$$m = 19 \, \text{kg}$$