Lecture

• Center of Mass (CM)
• Angular Quantities
• Torque
• Rotational Dynamics
• Angular Momentum*
A uranium nucleus (at rest) undergoes fission and splits into two fragments, one heavy and the other light. Which fragment has the greater momentum?

a) the heavy one

b) the light one

c) both have the same momentum

d) impossible to say
Center of Mass

In (a), the diver’s motion is pure translation; in (b) it is translation plus rotation.

Observation: There is one point that moves in the same path a particle would take if subjected to the same force as the diver.

This point is called the center of mass (CM).
Center of Mass

The general motion of an object can be considered as the sum of the translational motion of the CM, plus rotational, vibrational, or other forms of motion about the CM.
Center of Mass (1D)

For two particles, the center of mass lies closer to the one with the most mass:

\[ x_{CM} = \frac{m_A x_A + m_B x_B}{m_A + m_B} = \frac{m_A x_A + m_B x_B}{M} \]

where \( M \) is the total mass: \( M = m_A + m_B \).

This equation can be easily extended to many ("N") particles:

\[ x_{CM} = \frac{m_A x_A + m_B x_B + \ldots + m_N x_N}{m_A + m_B + \ldots + m_N} \]
Center of Mass (2D and 3D)

In more than one dimension we can simply find the CM for each independently:

\[ y_{CM} = \frac{m_A y_A + m_B y_B + \ldots + m_N y_N}{m_A + m_B + \ldots + m_N} \]

\[ z_{CM} = \frac{m_A z_A + m_B z_B + \ldots + m_N z_N}{m_A + m_B + \ldots + m_N} \]
Center of Mass vs. Center of Gravity

- The center of gravity is the point where the gravitational force can be considered to act.
- It is the same as the center of mass as long as the gravitational force does not vary among different parts of the object.
Center of Mass

The center of gravity can be found experimentally by suspending an object from different points.

The CM need not be within the actual object; a doughnut’s CM is in the center of the hole.
CM for the Human Body

The location of the center of mass of the leg (circled) will depend on the position of the leg.
CM for the Human Body

High jumpers have developed a technique where their CM actually passes under the bar as they go over it. This allows them to clear higher bars.
The disk shown below in (1) clearly has its center of mass at the center. Suppose the disk is cut in half and the pieces arranged as shown in (2). Where is the center of mass of (2) as compared to (1)?

a) higher  
b) lower  
c) at the same place  
d) there is no definable CM in this case
Center of Mass and Translational Motion

The CM (which may not correspond to the position of any particle) continues to move according to the net force.
Angular Quantities

• In purely rotational motion, all points on the object move in circles around the axis of rotation (“O”).

• The radius of the circle is $r$. All points on a straight line drawn through the axis move through the same angle in the same time.

The angle $\theta$ in radians is:

$$\theta = \frac{\ell}{r},$$

where $\ell$ is the arc length.
Angular Quantities

- **Angular displacement:** \( \Delta \theta = \theta_2 - \theta_1 \)

- The average angular velocity is defined as the total angular displacement divided by time:
  \[
  \omega_{\text{average}} = \frac{\Delta \theta}{\Delta t}
  \]

- The instantaneous angular velocity:
  \[
  \omega = \lim_{\Delta t \to 0} \frac{\Delta \theta}{\Delta t}
  \]
Bonnie sits on the outer rim of a merry-go-round, and Klyde sits midway between the center and the rim. The merry-go-round makes one complete revolution every two seconds. Who has the larger angular velocity?

a) Klyde  
b) Bonnie  
c) both the same  
d) angular velocity is zero for both of them
Angular Quantities

• The angular acceleration is the rate at which the angular velocity changes with time:

$$\alpha_{\text{average}} = \frac{\Delta \omega}{\Delta t}$$

• The instantaneous acceleration:

$$\alpha = \lim_{{\Delta t \to 0}} \frac{\Delta \omega}{\Delta t}$$
Angular Quantities

• Every point on a rotating body has an angular velocity $\omega = \frac{\Delta \theta}{\Delta t}$ and a linear velocity $\nu = \frac{\Delta \ell}{\Delta t}$.

• The arclength $\Delta \ell$ is related to the angular displacement $\Delta \theta$: $\Delta \ell = r \Delta \theta$

• Then $\nu$ and $\omega$ are related:

$$\nu = \frac{r \Delta \theta}{\Delta t} = r \omega$$
Angular Quantities

Therefore, objects farther from the axis of rotation will move faster.
Tangential and Centripetal Acceleration

• If the angular velocity of a rotating object changes, it has a tangential acceleration:

$$a_{\text{tan}} = \frac{\Delta v}{\Delta t} = r \frac{\Delta \omega}{\Delta t} = r \alpha$$

• Even if the angular velocity is constant, each point on the object has a centripetal acceleration:

$$a_c = \frac{v^2}{r} = \frac{(r \omega)^2}{r} = \omega^2 r$$
Angular Quantities

Here is the correspondence between linear and rotational quantities:

<table>
<thead>
<tr>
<th>Linear</th>
<th>Type</th>
<th>Rotational</th>
<th>Relation‡</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>displacement</td>
<td>$\theta$</td>
<td>$x = r\theta$</td>
</tr>
<tr>
<td>$v$</td>
<td>velocity</td>
<td>$\omega$</td>
<td>$v = r\omega$</td>
</tr>
<tr>
<td>$a_{\text{tan}}$</td>
<td>acceleration</td>
<td>$\alpha$</td>
<td>$a_{\text{tan}} = r\alpha$</td>
</tr>
</tbody>
</table>

‡ You must use radians.
Angular Quantities

Frequency is the number of complete revolutions per second:

\[ f = \frac{\omega}{2\pi} \]

Frequencies are measured in hertz.

\[ 1 \text{ Hz} = 1 \text{ s}^{-1} \]

The period is the time one revolution takes. It is related to frequency:

\[ T = \frac{1}{f} \]
Constant Angular Acceleration ($\alpha$)

The equations of motion for constant angular acceleration are the same as those for linear motion, with the substitution of the angular quantities for the linear ones.

Constant Angular Acceleration ($\alpha$)

\[
\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2
\]
\[
\omega = \omega_0 + \alpha t
\]
\[
\omega^2 = \omega_0^2 + 2\alpha \Delta\theta
\]
\[
\omega_{\text{average}} = \frac{\omega + \omega_0}{2}
\]

Constant Linear Acceleration ($a_x$)

\[
x = x_0 + v_{x_0} t + \frac{1}{2} a_x t^2
\]
\[
v_x = v_{x_0} + a_x t
\]
\[
v_x^2 = v_{x_0}^2 + 2a_x \Delta x
\]
\[
v_{x_{\text{average}}} = \frac{v_x + v_{x_0}}{2}
\]
Rolling Motion (Without Slipping)

In (a), a wheel is rolling without slipping. The point P, touching the ground, is instantaneously at rest, and the center moves with velocity \( v \).

In (b) the same wheel is seen from a reference frame where C is at rest. Now point P is moving with velocity \( -v \).
Vector Nature of Angular Quantities

• The angular velocity vector points along the axis of rotation; its direction is found using a right hand rule:
Vector (cross) Product

• The vector (or cross) product of two vectors is defined by:

\[
\hat{x} \times \hat{y} = \hat{z} \quad \hat{y} \times \hat{x} = -\hat{z} \\
\hat{y} \times \hat{z} = \hat{x} \quad \hat{z} \times \hat{y} = -\hat{x} \\
\hat{z} \times \hat{x} = \hat{y} \quad \hat{x} \times \hat{z} = -\hat{y}
\]

• zero otherwise.

• An important property of cross products is:

\[
|\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta
\]
Direction of Vector $\vec{C} = \vec{A} \times \vec{B}$

• The result of a vector cross product operation is itself a vector.

• The direction of the resulting vector is perpendicular to both vectors being cross-multiplied and given by the right-hand-rule.
You are using a wrench to loosen a rusty nut. Which arrangement will be the most effective in loosening the nut?

a) 

b) 

c) 

d) 

e) all are equally effective
To make an object start rotating, a force is needed.

The position and direction of the force matter as well.

The perpendicular distance from the axis of rotation to the line along which the force acts is called the lever arm ($\ell$).
Torque

A longer lever arm is very helpful in rotating objects.
Consider the wrench pivoted on the axis “O”.

If we apply a force $\vec{F}$ in a direction $\phi$ to the horizontal we can define the magnitude of the torque as:

$$\tau = rF \sin \phi$$

Torque should not be confused with force.

Forces can cause a change in linear acceleration (i.e. Newton’s 2\textsuperscript{nd} Law). Forces can also cause a change in rotation but the effectiveness of the forces depends on both the forces and the moment arms of the forces, in a combination that we call torque.
Torque

Magnitude of the torque is: $|\vec{\tau}| = |\vec{r}||\vec{F}|sin\theta$

If an object has zero net external torque then is said to be in “rotational” equilibrium:

$\Sigma \tau_{ext} = 0$

Then if both $\Sigma \vec{F}_{ext} = 0$ and $\Sigma \vec{\tau}_{ext} = 0$ the object is in equilibrium.
IClicker Question

Can an object have $\sum F_{\text{ext}} = 0$ and NOT be in equilibrium?

a) Yes

b) No
Two forces produce the same torque. Does it follow that they have the same magnitude?

a) Yes
b) No

ICLICKER QUESTION

c) Depends
Rotational Dynamics; Torque and Rotational Inertia

• Since $\Sigma |\vec{F}| = ma \Rightarrow |\vec{\tau}| = mra$ (F perpendicular to r), then for a point mass we can write: $|\vec{\tau}| = mr^2 \alpha$  

• This is for a single point mass; what about an extended object?

• Because the angular acceleration is the same for the whole object, we have:

$$\Sigma \tau = (\Sigma mr^2) \alpha$$
Rotational Dynamics; Torque and Rotational Inertia

The quantity $I = \Sigma mr^2$ is called the moment of inertia of an object.

The distribution of mass with respect to the axis of rotation matters!
Newton’s 2nd Law for Rotation

\[ \Sigma \ddot{\tau} = I \ddot{\alpha} \]

Where \( I \) is the moment of inertia of a mass distribution about a rotating axis. Notice that \( I \) analogous to mass for rotating objects.
The rotational inertia of an object depends not only on its mass distribution but also the location of the axis of rotation—compare (f) and (g), for example.
Two spheres have the same radius and equal masses. One is made of solid aluminum, and the other is made from a hollow shell of gold. Which one has the bigger moment of inertia about an axis through its center?

a) solid aluminum  
b) hollow gold  
c) same