Analytical Physics 1B Lecture 2: Angular Momentum

Sang-Wook Cheong, Friday, January 27th, 2018

Pls inform any issues with TA/PTL to me or Prof. Lath.

More time/make-up for exams (1st one: Feb 12): inform Prof. Lath (Letter of Accommodations (LOA) from the Office of Disability Services (ODS).

iClicker: not in the final grade
If you care about integration, then for a solid cylinder, we have: (ρ is density of the cylinder.)
Tesla S:
• 258 hp
• Torque: 317 lb.ft
• 0 to 60 mph time of 5.1 seconds

Mercedes-Benz E300
• 241 hp
• Torque: 273 lb.ft
• 0 to 60 mph time of 6.2 seconds

Nissan Centra
• 124 hp
• Torque: 125 lb.ft
• 0 to 60 mph time of 10 seconds
Tesla S:
• 258 hp
• Torque: $317 \text{ lb ft} = 317 \text{ lb ft} \times 0.454 \text{ kg/lb} \times 9.8 \text{ m/s}^2 \times 0.305 \text{ m/ft}$
• 0 to 60 mph time of 5.1 seconds

Mersedes-Benz E300
• 241 hp
• Torque: 273 lb ft
• 0 to 60 mph time of 6.2 seconds

Nissan Centra
• 124 hp
• Torque: 125 lb ft
• 0 to 60 mph time of 10 seconds
\[|\vec{\tau}| = |\vec{r}| |\vec{F}| \sin(\theta)\]
Torque

• In the simplest case, you will only be concerned about rotation around one axis
  • In this case, you can treat it like a scalar, where
    • it is **positive** when it corresponds to **CCW rotation**
    • it is **negative** when it corresponds to **CW rotation**

• In this case, you need to compute the magnitude of the torque

\[ |\vec{\tau}| = |\vec{r}| |\vec{F}| \sin(\theta) \]

• where \( \theta \) is the angle between \( r \) and \( F \)
Torque

• More generally, torque is defined as
  \[ \vec{\tau} = \vec{r} \times \vec{F} \]

• Torque is a vector*, so it adds vectorially

If you point the fingers of your right hand in the direction of \( \vec{r} \) and then curl them in the direction of \( \vec{F} \), your outstretched thumb points in the direction of \( \vec{\tau} \).

*Technically, torque is a pseudovector, not a vector, but don’t worry about this difference.
Newton’s Second Law Redux

• When you have a **rigid body** (this is an important assumption), then you get

\[ \sum \tau_z = I \alpha_z \]

• where the z axis is chosen to be the **axis of rotation** (not z direction)
• I is the moment of inertia
  • Don’t forget that this variable also depends implicitly on the axis of rotation as well

• **Notice the similarity to Newton’s second law**
  • What this means is that the sum of all torques about an axis determines the angular acceleration about that axis
A simplified Tesla S:

- Mass: 100 kg
- Force: 860 N
- Distance: 1 m
\[ \alpha = FR = 430 \text{ N} \cdot \text{m} = I \alpha = (\frac{1}{2} MR^2) \alpha \]

\[ \alpha = \frac{FR}{MR^2/2} = \frac{2F}{mR} = \frac{2.860 \text{ N}}{100 \text{ kg} \cdot 0.5 \text{ m}} = 34.4 \text{ rad/s}^2 \]

\[ \omega_2 = \alpha t = 0.5 \text{ m} \cdot 34.4 \text{ rad/s}^2 \cdot 1 \text{ s} = 17.2 \text{ m/s}^2 \]

\[ \delta = \alpha_2 t^2 = 17.2 \times 6 \text{ m/s} = 103.2 \text{ m/s} \]

\[ = 103.2 \text{ m/s} \cdot \frac{3600 \text{ s}}{1 \text{ h}} \cdot \frac{1 \text{ mile}}{1609 \text{ m}} \]

\[ = 221 \text{ mile/h} \]

\[ P = \gamma \omega_2 = 430 \text{ N} \cdot \text{m} \cdot 34.4 \text{ rad/s}^2 \cdot 1 \text{ s} \]

\[ = 8.88 \times 10^4 \text{ J/s} = 8.88 \times 10^4 \text{ W} \]

\[ 1 \text{ horsepower} = 745.7 \text{ W} \]

\[ = 119 \text{ hp} \]
Combining Energies

• The motion of a **rigid body** can always be separated into
  • a translation of the center of mass
  • a rotation about the center of mass

\[
K = \frac{1}{2} M v_{\text{cm}}^2 + \frac{1}{2} I_{\text{cm}} \omega^2
\]

  translational component  angular component

• For a rigid body, kinetic energy has **two components**
Demo on Rolling Cylinders

General equation describing the energy of a rolling cylinder:

\[ E = \frac{1}{2} M v_{\text{cm}}^2 + \frac{1}{2} I_{\text{cm}} \omega^2 + Mgh \]

Which is faster?

Solid cylinder | Hollow cylinder
---|---
B | A

Zero level \( h_f = 0 \text{ m} \)

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Physics 124 – Angular Momentum
Rolling Without Slipping

- The condition for rolling without slipping is $v_{cm} = R \omega$. 

Translation of center of mass:
velocity $v_{cm}$

Rotation around center of mass:
for rolling without slipping, speed at rim = $v_{cm}$

Combined motion

Wheel is instantaneously at rest where it contacts the ground.
\[ M \mathbf{\dot{\theta}} = \frac{1}{2} m \mathbf{\dot{u}}^2 + \frac{1}{2} I \left( \frac{\mathbf{\dot{u}}}{R} \right)^2 \]

\[ = \frac{1}{2} m \mathbf{\dot{u}}^2 + \frac{1}{2} \left( \frac{1}{2} mR^2 \right) \left( \frac{\mathbf{\dot{u}}}{R} \right)^2 = \frac{3}{4} M \mathbf{\dot{u}}^2 \]

\[ \mathbf{\dot{u}} = \sqrt{\frac{4gh}{3}} \]

\[ = \frac{1}{2} m \mathbf{\dot{u}}^2 + \frac{1}{2} mR^2 \left( \frac{\mathbf{\dot{u}}}{R} \right)^2 = M \mathbf{\dot{u}}^2 \]

\[ \mathbf{\dot{u}} = \sqrt{gh} \]
Work and Power

• The work done by a torque about a fixed axis $z$ is

$$W = \int_{\theta_1}^{\theta_2} \tau_z d\theta$$

• For a **rigid body**:  

$$W_{\text{tot}} = \frac{1}{2} I \omega_2^2 - \frac{1}{2} I \omega_1^2$$

$$P = \frac{dW}{dt} = \tau_z \omega_z$$

Remember the analogous equation for work done by a force
A simplified Tesla S:

\[ 860 \text{ N} \]

\[ 100 \text{ kg} \]

\[ 1 \text{ m} \]
\[ \tau = FR = 430 \text{ N} \cdot \text{m} = I \alpha_2 = \left( \frac{1}{2} MR^2 \right) \alpha_2 \]

\[ \alpha_2 = \frac{FR}{mR^2/2} = \frac{2F}{mR} = \frac{2 \cdot 860 \text{ N}}{100 \text{ kg} \cdot 0.5 \text{ m}} = 34.4 \text{ rad/s}^2 \]

\[ \omega_2^2 = \alpha_2 \cdot t \]

\[ \omega_{\text{tan}} = R \omega_2 = 0.5 \text{ m} \cdot 34.4 \text{ rad/s} = 17.2 \text{ m/s} \]

\[ \omega_{\text{tan}} \cdot t = 17.2 \times 6 \text{ m/s} = 103.2 \text{ m/s} \]

\[ \omega_{\text{tan}} = \frac{103.2 \text{ m/s}}{3600 \text{ s}} = \frac{m \cdot \text{sec}}{1609 \text{ m}} \]

\[ = 0.06 \text{ mile/e} \]
Tesla S:
- 258 hp
- Torque: 317 lb.ft = 430 N.m
- 0 to 60 mph time of 5.1 seconds

Mersedes-Benz E300
- 241 hp
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Nissan Centra
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Angular Momentum

- Angular momentum is defined as
  \[ \vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{\dot{v}} \]

- Taking the derivative of L with respect to time, we find:
  \[ \frac{d\vec{L}}{dt} = \left( \frac{d\vec{r}}{dt} \times m\vec{\dot{v}} \right) + \left( \vec{r} \times m \frac{d\vec{\dot{v}}}{dt} \right) = (\vec{\dot{v}} \times m\vec{\dot{v}}) + (\vec{r} \times m\vec{\ddot{a}}) \]
  - The first term vanishes, and the second term is just the torque!
  \[ \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \]
Angular Momentum

- As with torque, one often deals the magnitude of the angular momentum:

\[ \vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \]

\[ |\vec{L}| = |\vec{r}| |\vec{p}| \sin \theta = |\vec{r}| m |\vec{v}| \sin \theta \]

- where \( \theta \) is the angle between \( \vec{r} \) and \( \vec{p} \)

- For a rigid body

\[ \vec{L} = I \vec{\omega} \]
Angular Momentum Conservation

• When the net external torque acting on a system is zero, the total angular momentum of the system is constant (conserved)
  • You now know the three main conservation laws:
    • energy
    • momentum
    • angular momentum

• Identifying when a given (sub)system has no net torque applied will be critical to solving problems
Angular Momentum Conservation

Dumbbell

Professor (not a dumbbell)

Before

After

$\omega_1$

$\omega_2$
Angular Momentum Conservation

5 kg dumbbell
I of professor Before = 3 kg.m²
I of professor After = 2.2 kg.m²

2 seconds for one revolution
\[ I_1 = 3 \text{ kg} \cdot \text{m}^2 + 2(5 \text{ kg})(1 \text{ m})^2 = 13 \text{ kg} \cdot \text{m}^2 \]

\[ \omega_{12} = \frac{1 \text{ rev}}{2 \text{ s}} = 0.5 \text{ rev/s} \]

\[ I_2 = 2.2 \text{ kg} \cdot \text{m}^2 + 2(5 \text{ kg})(0.2 \text{ m})^2 = 2.6 \text{ kg} \cdot \text{m}^2 \]

\[ \omega_{23} = \frac{I_1}{I_2} \omega_{12} = \frac{13 \text{ kg} \cdot \text{m}^2}{2.6 \text{ kg} \cdot \text{m}^2} (0.5 \text{ rev/s}) = 2.5 \text{ rev/s} = 5 \omega_{12} \]

\[ K_1 = \frac{1}{2} I_1 \omega_{12}^2 = \frac{1}{2} (13 \text{ kg} \cdot \text{m}^2)(0.5 \text{ rev/s})^2 = 64 \text{ J} \]

\[ K_2 = \frac{1}{2} I_2 \omega_{23}^2 = \frac{1}{2} (2.6 \text{ kg} \cdot \text{m}^2)(5 \text{ rev/s})^2 = 320 \text{ J} \]
Why do cats *always* land upside up, but pizza land upside down?
Why do cats *always* land on their feet?
Why do cats **always** land on their feet?
Cat Videos

• Do cats **always** land on their feet?
  
  • **What if you drop them upside down?**
    • If no torque is applied initially, the law of conservation of angular momentum would tell you that they **cannot** land on their feet.

• [https://www.youtube.com/watch?v=RtWbpyjJqrU](https://www.youtube.com/watch?v=RtWbpyjJqrU)
• [https://www.youtube.com/watch?v=yGusK69XVlk](https://www.youtube.com/watch?v=yGusK69XVlk)