Mathematical Description of a Wave

• From last week we saw that a sinusoidal wave traveling left to right is described by

\[ y(x, t) = A \cos(kx - \omega t) \]

• where \( k \) a quantity we define as the wave number

\[ k \equiv \frac{2\pi}{\lambda} \]

• For a wave travelling right to left:

\[ y(x, t) = A \cos(kx + \omega t) \]
Wave Equation

- Take partial derivatives of the wave equation with respect to time
  - (a partial derivative is just a derivative where you treat all but one variable constant)

\[
y(x, t) = A \cos(kx - \omega t)
\]

Vertical position of a particle on a wave as a function of x and t

\[
\frac{\partial}{\partial t} y(x, t) = \omega A \sin(kx - \omega t)
\]

Transverse velocity of a particle on a wave as a function of x and t

\[
\frac{\partial^2}{\partial t^2} y(x, t) = -\omega^2 A \cos(kx - \omega t) = -\omega^2 y(x, t)
\]

Transverse acceleration of a particle on a wave as a function of x and t
Wave Equation

- Take partial derivatives of the wave equation with respect to position

\[ y(x, t) = A \cos(kx - \omega t) \]

Vertical position of a particle on a wave as a function of x and t

\[ \frac{\partial}{\partial x} y(x, t) = -Ak \sin(kx - \omega t) \]

Slope of the wave as a function of x and t

\[ \frac{\partial^2}{\partial x^2} y(x, t) = -Ak^2 \cos(kx - \omega t) = -k^2 y(x, t) \]

Curvature of the wave as a function of x and t
Wave Equation

- Notice that the second partial derivatives are related to each other:

\[
\frac{\partial^2 y(x, t)}{\partial x^2} = -k^2 y(x, t) \quad \quad \quad \quad \quad \frac{\partial^2 y(x, t)}{\partial t^2} = -\omega^2 y(x, t)
\]

- Also notice that the velocity of the wave is related to the frequency and the wave number

\[
v = \frac{\omega}{k}
\]

- Therefore:

\[
\frac{\partial^2 y(x, t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y(x, t)}{\partial t^2}
\]

This is the “wave equation”
The wave equation says that the curvature of the wave at a given point is proportional to the acceleration of that point.

- Notice also that when the curvature is pointing upward, the acceleration is also pointing upward.
A Gaussian Wave

• The wave equation is **way more general** than just a sinusoidal wave
  • Consider the following function: \[ y(x, t) = e^{-\frac{(x-vt)^2}{2}} \]
    • This is a “Gaussian” wave
      • (if you want an exercise in partial differentiation, take this equation, and show that it satisfies the wave equation)

• This is a **single pulse** traveling left to right
• four snapshots are shown at t=0, 1, 2, 3 sec
• The velocity is 1 m/sec
Wave Speed

• Waves can travel at different speeds
  • light in a vacuum travels about 1 billion feet in one second
  • sound in room temperature air travels about 1 thousand feet in one second

• The speed of a mechanical wave depends on two things
  • The strength of the restoring force that brings the system back to equilibrium
  • The inertia of the system resisting the return to equilibrium
Wave Speed

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- The speed of a mechanical wave depends on two things
  - The strength of the restoring force that brings the system back to equilibrium
  - The inertia of the system resisting the return to equilibrium

- For a string these are:
  - the tension on the string, $F$
  - mass per unit length of the string, $\mu$

\[ v = \sqrt{\frac{F}{\mu}} \]
Two pulses travel toward each other along a long stretched spring as shown. Pulse A is wider than pulse B, but not as high.

What is the speed of pulse A compared to pulse B?

The speed of pulse A is ___________ the speed of pulse B.

A.) Equal to
B.) Larger than
C.) Smaller than
D.) Not enough information
Pulse Speed

Two pulses travel toward each other along a long stretched spring, with uniform density, as shown. Pulse A is wider than pulse B, but not as high.

What is the speed of pulse A compared to pulse B?

The speed of pulse A is ___________ the speed of pulse B.

A.) Equal to  
B.) Larger than  
C.) Smaller than  
D.) Not enough information
**Boundary Conditions**

(a) Wave reflects from a fixed end.

1. Pulse arrives.
2. String exerts an upward force on wall ...
3. ... wall exerts a downward reaction force on string.
4. Pulse inverts as it reflects.

(b) Wave reflects from a free end.

1. Pulse arrives.
2. Rod exerts no transverse forces on string.
3. Pulse reflects without inverting.
What picture best describes what the pulse on the string looks like after it has completely reflected from the wall.

A.)  

B.)  

C.)  

D.)  

E.)  Not enough information.
A long string is firmly connected to a stationary metal rod at one end. A student holding the other end moves her hand rapidly up and down to create a pulse towards the rod.

What picture best describes what the pulse on the string looks like after it has completely reflected from the wall.

**Standing Waves**

- Standing waves are created when a sinusoidal wave is reflected back by the fixed end of a string and interfere with each other.

(a) String is one-half wavelength long.  
(b) String is one wavelength long.  
(c) String is one and a half wavelengths long.  
(d) String is two wavelengths long.  
(e) The shape of the string in (b) at two different instants.

\[ N = \text{nodes}: \text{points at which the string never moves} \]

\[ A = \text{antinodes}: \text{points at which the amplitude of string motion is greatest} \]
STANDING WAVES

Consider a sinusoidal wave that is traveling left to right and is reflected back against a stationary point with an identical wave traveling right to left in phase.

\[
y_1(x, t) = A \cos(kx - \omega t) \quad \text{wave moving left to right}
\]

\[
y_2(x, t) = -A \cos(kx + \omega t) \quad \text{wave moving right to left (upside down)}
\]

\[
y(x, t) = y_1(x, t) + y_2(x, t) \quad \text{principle of superposition}
\]

\[
y(x, t) = A [\cos(kx - \omega t) - \cos(kx + \omega t)]
\]

\[
y(x, t) = A [\cos(kx) \cos(\omega t) - \sin(kx) \sin(\omega t) - \cos(kx) \cos(\omega t) - \sin(kx) \sin(\omega t)]
\]

\[
y(x, t) = 2A \sin(kx) \sin(\omega t)
\]
STANDING WAVES

\[ y(x, t) = 2A \sin(kx) \sin(\omega t) \]

- At every instance, \( y(x,t) \) is sinusoidal
  - But there is no **traveling**: the shape stays in the same position, oscillating up and down
  - Unlike in a traveling wave, **all points oscillate in phase**
  - There is no transfer of energy from one part of the string to another

- Nodes occur when
  \[ kx = 0, \pi, 2\pi, 3\pi, \ldots \]
  - Or equivalently, when
  \[ x = 0, \frac{\lambda}{2}, \frac{2\lambda}{2}, \frac{3\lambda}{2}, \ldots \]
**Fundamental Frequencies**

- Suppose you fix a string at both ends
  - A standing wave will occur with wavelengths
    \[ \lambda_n = \frac{2L}{n} \quad (n = 1, 2, 3, \ldots) \]
  - equivalently, the frequencies will have the values
    \[ f_n = n \frac{v}{2L} \quad (n = 1, 2, 3, \ldots) \]
- \( f_1 \) is called the **fundamental frequency**
Standing Waves for Strings

• For a mechanical wave, we know that

\[ v = \sqrt{\frac{F}{\mu}} \]

• It follows then that the standing waves occur at frequencies

\[ f_n = n \frac{1}{2L} \sqrt{\frac{F}{\mu}} \quad (n = 1, 2, 3, \ldots) \]
A string is stretched so that it is under tension and is tied at both ends so that the endpoints don’t move. A mechanical oscillator then vibrates the string so that a standing wave is created. All of the strings have the same length but may not have the same mass. The number of nodes and antinodes in the standing wave is the same in Cases A and D. The tensions in the strings ($T$) and the standing wave frequencies ($f$) are given in each figure.

Rank the speeds of the wave in the string.

A.) $A > D > C > B$
B.) $B > C > D > A$
C.) $C > B = A > D$
D.) $C = B > A = D$
E.) The speeds are the same for all cases
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