Analytical Physics 1B Lecture 2: Angular Momentum

John Paul Chou
\[ |\vec{\tau}| = |\vec{r}| |\vec{F}| \sin(\theta) \]
TORQUE

• In the simplest case, you will only be concerned about rotation **around one axis**
  
  • In this case, you can treat it like a scalar, where
    • it is **positive** when it corresponds to **CCW rotation**
    • it is **negative** when it corresponds to **CW rotation**

  • In this case, you need to compute the **magnitude of the torque**

  \[
  |\vec{\tau}| = |\vec{r}| |\vec{F}| \sin(\theta)
  \]

  • where $\theta$ is the angle between $\vec{r}$ and $\vec{F}$
TORQUE

• More generally, torque is **defined as**

\[ \vec{\tau} = \vec{r} \times \vec{F} \]

• Torque is a vector, so it adds **vectorially**

If you point the fingers of your right hand in the direction of \( \vec{r} \) and then curl them in the direction of \( \vec{F} \), your outstretched thumb points in the direction of \( \vec{\tau} \).
NEWTON’S SECOND LAW REDUX

• When you have a **rigid body** (this is an important assumption), then you get

\[ \sum \tau_z = I \alpha_z \]

• where the z axis is chosen to be the **axis of rotation** (not z direction)
• I is the moment of inertia
  • Don’t forget that this variable also depends implicitly on the axis of rotation as well

• **Notice the similarity to Newton’s second law**
  • What this means is that the sum of all torques about an axis determines the angular acceleration about that axis
In each figure below, the jet engine is slowing down due to the application of a constant torque. All of the engines are identical, but they start with different angular speeds and have torques of different magnitudes applied to the rotating shafts within the engines. Magnitudes of the initial angular speeds and torques are given in the figures.

Rank these situations on the basis of the magnitude of the angular acceleration of the engines as they slow down, from GREATEST to LEAST.

1. B > D > E > F > A > C
2. C > A > F > E > D > B
3. C = D = F > A = B = E
4. B = D > E = F > A = C
5. A = B = C = D = E = F
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1. $B > D > E > F > A > C$
2. $C > A > F > E > D > B$
3. $C = D = F > A = B = E$
4. $B = D > E = F > A = C$
5. $A = B = C = D = E = F$

Angular acceleration is determined by the torque, so ordering is determined solely by $\tau$. 
Combining Energies

• The motion of a **rigid body** can always be separated into
  • a translation of the center of mass
  • a rotation about the center of mass

\[
K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2
\]

  translational component  angular component

• For a rigid body, kinetic energy has **two components**
**Demo on Rolling Cylinders**

General equation describing the energy of a rolling cylinder:

\[ E = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2 + Mgh \]

Which is faster?

Initially, there is no velocity:

\[ E_{init} = Mgh \]

Finally, there is no potential energy:

\[ E_{final} = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2 \]

This is the condition for **rolling without slipping**:

\[ v_{cm} = R\omega \]

Conservation of energy means \( E_{init} = E_{final} \).

Equating these two and performing some algebra (confirm this for yourself at home) gives…
**Demo on Rolling Cylinders**

- The final velocity of the object

\[ v_{\text{cm}} = \sqrt{\frac{2ghMR^2}{MR^2 + I_{\text{cm}}}} \]

Which is faster?

- if we compare two objects, the one with the smaller \( I_{\text{cm}} \) will have the higher final velocity

- Solid cylinder: \( I_{\text{cm}} = MR^2/2 \)
- Hollow cylinder: \( I_{\text{cm}} = MR^2 \)

- Prediction: a solid cylinder will beat a hollow cylinder down a ramp
  - does the mass or radius matter?
**Work and Power**

- The work done by a torque about a fixed axis $z$ is

$$W = \int_{\theta_1}^{\theta_2} \tau_z d\theta$$

Remember the analogous equation for work done by a force

$$(W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{l})$$

- For a **rigid body**:

$$W_{\text{tot}} = \frac{1}{2} I \omega_2^2 - \frac{1}{2} I \omega_1^2$$

How much work is done to keep a propeller spinning at a constant speed?
Angular Momentum

• Angular momentum is defined as

\[ \vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} \]

• Taking the derivative of L with respect to time, we find:

\[ \frac{d\vec{L}}{dt} = \left( \frac{d\vec{r}}{dt} \times m\vec{v} \right) + \left( \vec{r} \times \frac{d\vec{v}}{dt} \right) = (\vec{v} \times m\vec{v}) + (\vec{r} \times m\vec{a}) \]

• The first term vanishes, and the second term is just the torque!

\[ \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{\tau} \]

• Does this formula remind you of another one?
As with torque, one often deals the *magnitude* of the angular momentum:

\[
\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}
\]

\[
|\vec{L}| = |\vec{r}| |\vec{p}| \sin \theta = |\vec{r}| m |\vec{v}| \sin \theta
\]

- where \( \theta \) is the angle between \( \vec{r} \) and \( \vec{p} \)

- For a *rigid body*

\[
\vec{L} = I \vec{\omega}
\]
Angular Momentum Conservation

• When the net external torque acting on a system is zero, the total angular momentum of the system is constant (conserved)
  • You now know the three main conservation laws:
    • energy
    • momentum
    • angular momentum

• Identifying when a given (sub)system has no net torque applied will be critical to solving problems
Cat Videos

• Do cats **always** land on their feet?

  • **What if you drop them upside down?**
    • If no angular momentum is applied initially, the law of conservation of angular momentum would tell you that they **cannot** land on their feet.

  

• [https://www.youtube.com/watch?v=RtWbpyjJqrU](https://www.youtube.com/watch?v=RtWbpyjJqrU)
• [https://www.youtube.com/watch?v=yGusK69XVlk](https://www.youtube.com/watch?v=yGusK69XVlk)
A weight is tied to a rope that is wrapped around a pulley. The pulley is initially rotating counterclockwise and is pulling the weight up. The tension in the rope creates a torque on the pulley that opposes this rotation. Consider the following statements which MIGHT be true at the instant the pulley stops rotating counterclockwise before starting to rotate clockwise:

A. The torque on the pulley about its axis is equal to zero.
B. The angular acceleration of the pulley is equal to zero.
C. The angular momentum of the pulley about the axis is equal to zero.

Which of the following is(are) TRUE?

1. A only
2. B only
3. C only
4. A and B
5. A, B, and C
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3. C only
4. A and B
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There is always torque on the pulley because gravity is always pulling on the weight. \( \tau = I \alpha \), so non-zero \( \tau \) gives non-zero \( \alpha \). So statements A and B are false. Statement C is true because \( L = I \omega \), so if \( \omega = 0 \), \( L = 0 \).
An airplane is flying at a constant horizontal velocity at altitude $h$ above an observer. The observer on the ground measures the angle between the airplane and the horizontal at four different locations as shown. Rank the magnitude of the angular momentum of the airplane at the various locations relative to the ground observer, from GREATEST to LEAST.

1. A, B, C, D
2. A, C, D, B
3. A=B=C=D
4. B, D, C, A
5. None of these is correct
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2. A, C, D, B
3. $A=B=C=D$
4. B, D, C, A
5. None of these is correct

There is no net external torque on the airplane. $\frac{dL}{dt}=\tau=0$, so there is no change in the angular momentum in the four panes. $A=B=C=D$. 
A bullet of mass \( m \) is shot at a hinged rod and hits the rod a distance \( d \) from the hinge. The rod was initially at rest and has a moment of inertia of \( I_o \) about an axis through its hinge. The bullet is fired at an angle \( \theta \) to the rod, as shown in the top view, with an initial velocity of \( v \) at a distance \( d \) from the hinge. The angular speed of the rod with the bullet embedded right after the collision is \( \omega_f \). Assume there is no friction in the hinge and that the rod is free to rotate about the hinge axis. Consider the following two statements:

**A.** The angular momentum about the hinge of the rod and bullet together after the bullet is embedded is the same as the initial angular momentum of the bullet about the hinge.

**B.** The total kinetic energy of the rod and bullet after the bullet is embedded is the same as the initial kinetic energy of the bullet.

1. A only
2. B only
3. both A and B
4. There is not enough information to determine.
**BULLET SHOT AT ROD (iCLICKEr)**

A bullet of mass $m$ is shot at a hinged rod and hits the rod a distance $d$ from the hinge. The rod was initially at rest and has a moment of inertia of $I_o$ about an axis through its hinge. The bullet is fired at an angle $\theta$ to the rod, as shown in the top view, with an initial velocity of $v$ at a distance $d$ from the hinge. The angular speed of the rod with the bullet embedded right after the collision is $\omega_f$. Assume there is no friction in the hinge and that the rod is free to rotate about the hinge axis. Consider the following two statements:

This is an inelastic scatter, so kinetic energy is not conserved, but (angular) momentum is. Therefore the angular momentum of the bullet+rod system is conserved throughout. Before the collision, the rod has zero $L$, so the angular momentum of the bullet before the collision is the same as the angular momentum of the bullet+rod system afterwards.

A. The angular momentum about the hinge of the rod and bullet together after the bullet is embedded is the same as the initial angular momentum of the bullet about the hinge.

B. The total kinetic energy of the rod and bullet after the bullet is embedded is the same as the initial kinetic energy of the bullet.

1. A only
2. B only
3. both A and B
4. There is not enough information to determine.